Combinatorial suspension for disc-homeomorphisms

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December 14, 2000

Abstract: *For a punctured disc homeomorphism given combinatorially, we give an algorithmic construction of the suspension flow in the corresponding mapping-torus M^3. In particular, one computes explicitly the embedding in the 3-manifold M^3 of any finite collection of periodic orbits for this flow. All these orbits are realized as closed braids carried by a branched surface (or template), which we construct in the algorithm. Our construction gives a combinatorial proof of the fact that the periodic orbits of such a suspension flow are carried by a same branched surface.*

Introduction

Among the many ideas and methods that H.Poincaré has introduced in his study of dynamical systems and topology, perhaps the most celebrated one was the notion of "local section" and "local return map". It has also been known for a very long time that the inverse operation, the so called *suspension flow* or *mapping torus*, is a well defined operation which is a very useful tool for instance to construct 3-dimensional manifolds. Going back to Poincaré he also gave the advice to study periodic orbits first if one wants to understand some of the global structure of a dynamical system.
Although the suspension is very easy to define, it is not that easy to construct in practice. Our goal in this paper is to construct a combinatorial version of the suspension flow, from a combinatorial description of a surface homeomorphism. Here the word construct has to be understood in the sense of an explicit algorithm. Let us be precise about the motivations of this work. If $f : S \to S$ is an orientation preserving surface homeomorphism, let Φ_t be a suspension flow defined by f on the 3-manifold M^3_f (the mapping torus). Every finite collection of periodic orbits P of f gives rise to a link L_P in M^3_f. In many cases the periodic orbits of f are described by a symbolic coding, for instance when a Markov partition is available (e.g. Axiom A, pseudo-Anosov,...). From this symbolic coding we should be able, in principle, to describe the embedding of the links L_P in 3 space. The goal of the paper is to turn this principle into an explicit description via a finite algorithm.
In order to be more explicit we are going to restrict the study to the homeomorphisms of the punctured disc. The combinatorial description of the homeomorphism is given as a *topological representative*, i.e. as a continuous map on a graph embedded in the surface. This special class of maps have been of central importance in the *train track algorithms* ([BH1], [FM], [Lo1], [Lo2]). In fact we restrict the study to the *efficient representatives* as

obtained at the end of the above algorithms since they carry all of the combinatorial informations about the canonical representative with respect to the Nielsen-Thurston theorem (see [Th1], [BH1], [Lo2]). The basic notions about efficient representatives are reviewed in section 1, as well as some other needed tools.

Since we only consider punctured disc homeomorphisms, the corresponding 3 manifold is the complement of a closed braid $\overline{\beta}$ in the solid torus. The suspension flow Φ_t is transverse to the meridian discs and all the periodic orbits of the flow are closed braids in the solid torus. If the homeomorphism is pseudo-Anosov (see [Th1], [FLP]), we also know that the number of periodic orbits is minimal in the isotopy class ([AF]). In fact the pseudo-Anosov representative posseses exactly the periodic orbit "types" (braid types or patterns) of the *genealogy* (see [Lo2]). The efficient representatives enables one to describe each periodic orbit of the pseudo-Anosov homeomorphism by an explicit symbolic description. Our goal is to transform this symbolic coding of a collection of periodic orbits P for the homeomorphism to a braid presentation of the link L_P. In particular, the linking information which is of central interest in several recent works (see for instance [GST]) is immediately computable.

The transition from the surface homeomorphism to the flow in 3-space is obtained via a special class of *branched surface* as first defined by R.F.Williams ([W73]). These branched surfaces have been studied from different points of views, for instance by J.Christy under the name of *dynamic branched surfaces* ([C]) and also under the name of of *knot holders* or *templates* by Birman-Williams ([BW]) or Ghrist-Holmes-Sullivan ([GHS]).

The special class we construct in this paper is called *braided branched surfaces*. They have the property of being embedded in the solid torus transversally to the meridian discs and their branched locus is a finite collection of intervals embedded in a single fiber. The complement of the branched locus is a collection of rectangles. For a given pseudo-Anosov isotopy class in the punctured disc $D_N = D^2 - N$ points induced by a braid $\beta \in \mathcal{B}_N$, it is proved in [Lo2] that there exists a braided branched surface $W_{\overline{\beta}}$ which carries, up to a finite number, the suspension of all the periodic orbits of the pseudo-Anosov representative in this class. Furthermore there is an explicit one to one map between a symbolic coding of the orbits of the pseudo-Anosov representative and the geometric realization of these orbits as closed braids carried by this *special* braided branched surface.

In sections 2 and 3 we give the algorithmic construction of such a braided branched surface starting from an efficient representative of the isotopy class. At the end of section 3 we give the description of the periodic orbits carried by the braided branched surface as closed braids. This description uses a variation of the kneading theory as defined in particular by Collet and Eckman [CE]. The main result of the paper can be stated as follows:

Theorem 0.1 *Let β be a braid in B_N whose corresponding isotopy class $[\varphi_\beta]$ is pseudo-Anosov. Let $\overline{\beta}$ be the closure of β in the solid torus* $\mathbf{T}$ *and let $M^3_{\overline{\beta}} = \mathbf{T}$ - $\overline{\beta}$ be the complement of the closed braid in the solid torus. Then there exists a finite algorithm starting from any induced automorphism $\varphi : \pi_1(D_N) \to \pi_1(D_N)$ which enables one to construct a special braided branched surface $W_{\overline{\beta}}$ in $M^3_{\overline{\beta}}$ which carries, up to a well defined finite collection, the suspension of all the periodic orbits of a pseudo-Anosov homeomorphism in the class $[\varphi_\beta]$.*

In fact our construction does not depend on the properties of the pseudo-Anosov representative in the class. It works also for all the homeomorphisms of the punctured disc which can be described by a topological representative. Another remark is that our description enables one to get, in fact, an embedding of a Markov partition of a suspension flow and not only the embedding of the periodic orbits. If the goal were just to obtain some suspension flow in a given isotopy class then our construction could be simplified by choosing a particular topological representative (see section 4).

1 Preliminaries

1.1 Graphs

A *graph* Γ is an oriented 1 dimensional CW-complex. The 0-cells (resp. 1-cells) are the *vertices* (resp. *edges*) of the graph and we denote by $V(\Gamma)$ (resp. $E(\Gamma)$) the set of vertices (resp. edges). For all the standard terminology about graphs such as: *valency (val(v)) and star (St(v)) of a* vertex, *edge path* (written as words in the letters $e_i^{\pm 1}$), *reduced edge path, initial (i(e))* and *terminal (t(e))* vertex of an edge (or an edge path), *length l(w)* of an edge path .... we refer the reader to [BH1], [Lo3] for instance. We call *subdivision* the operation which consists of declaring a point x in the interior of an edge e as a new vertex (of valency two).
A *tree* is a contractible graph or, equivalently, a graph such that any two points are connected by a unique reduced path. A tree is *oriented toward a vertex* v if the unique reduced path from any vertex $v_i \neq v$ to v is positive i.e. the letters occuring in the edge path have positive exponent. We distinguish among the edges incident at v_i (in $St(v_i)$) the *incoming edges* and the *outgoing edges* e_j, as those such that respectively $t(e_j) = v_i$ and $i(e_j) = v_i$ hold. In a tree oriented toward a vertex v, the vertex v is the only one such that $St(v)$ has no outgoing edges and, for all other vertices, there is exactly one outgoing edge.

In what follows all the graphs we consider are embedded in an orientable surface with boundary. Furthermore the graph and the surface have isomorphic fundamental groups, where the isomorphism is induced by the embedding of the graph in the surface. If a graph Γ is embedded in a surface S then, at each vertex the orientation of the surface induces a *cyclic ordering* of the incident edges. To a graph Γ embedded in a surface S, we associate a *fibered neighborhood* $N(\Gamma)$ in the following way:

- Around each vertex v of valency k, we define a k-gon $P(v)$,
- For each edge e, we define a rectangle $R(e)$,

and we glue them together in the surface, respecting the cyclic ordering, in the way illustrated by figure 1.

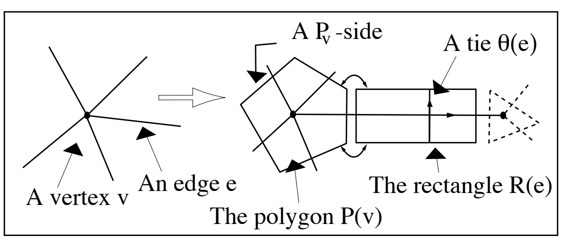


Figure 1: The fibered neighborhood $N(\Gamma)$

The graph Γ is embedded in the fibered neighborhood $N(\Gamma)$ according to the following construction. We parametrize the rectangles $R(e)$ by $[0,1]\times[-1,1]$ and $e \cap R(e) = [0,1]\times\{0\}$. The orientation is such that e and the oriented segment $\{x\}\times[-1,1]$ form a direct basis with respect to the orientation of the surface S. A *tie* $\theta(e)$ in a rectangle $R(e)$ is any oriented segment $\{x\}\times[-1,1]$, where $x \in Int(e \cap R(e))$. Each polygon $P(v)$ is a topological disc embedded in the surface which contains the vertex v of Γ in its interior. Its boundary $\partial P(v)$ is oriented according to the given cyclic ordering at the vertex v. The ties in a polygon $P(v)$ are the segments joining its boundary to the vertex v. Notice that there is a natural retraction r of $N(\Gamma)$ onto Γ by declaring two points equivalent if they belong to the same tie. The segments on the boundary of the polygons $P(v)$ are called the *P_v-sides*. The P_v-sides which intersect the incoming (resp. outgoing) edges will be called *incoming* (resp. *outgoing*).

A *turn at a vertex* v is an unordered pair of edges (e_i, e_j) in $St(v)$, and an *oriented turn* is a turn where the order of the pair is specified. A *direct turn* is an oriented turn such that the two edges are consecutive in the cyclic ordering at v.
A *path* π *in* $N(\Gamma)$ is an embedding of $[0,1]$ in $N(\Gamma)$ transverse to the ties. Observe that the end points of a path are not required to belong to a polygon $P(v)$. If we apply the retraction r to a path π in $N(\Gamma)$ we get an immersed path in Γ where the extrem points of $r \circ \pi$ are not necessarily vertices of Γ. By subdividing Γ at $\partial(r \circ \pi)$ (if necessary) we get a new graph Γ' and $r \circ \pi$ is now an edge path in Γ'. The corresponding word representing the edge-path $r \circ \pi$ in Γ' will be called a *longitudinal word*.

Definition 1.1 Let Γ be a graph embedded in a surface S, $N(\Gamma)$ its fibered neighborhood and let e be an edge of Γ. If $\pi = \{\pi_1, \cdots, \pi_k\}$ is a set of disjointly embedded paths in $N(\Gamma)$, let I and I' be two connected components of $\pi \cap R(e)$. Then we denote $I \prec_e I'$ if and only if the intersection point $I \cap \theta(e)$ follows the intersection point $I' \cap \theta(e)$ along the oriented tie $\theta(e)$.
Let $\{I_j\}_{j=1,\cdots,l}$ be the set of oriented segments of $\pi \cap R(e)$. We define the *transversal word* at $R(e)$ of the set π as:

$$R_\pi^\perp(e) = I_1^{\epsilon_1} I_2^{\epsilon_2} \cdots I_l^{\epsilon_l}$$

where the indices are so that $I_j \prec_e I_{j+1}$ and $\epsilon_i = +1$ if the orientation of I_i and e agree, $\epsilon_i = -1$ otherwise.

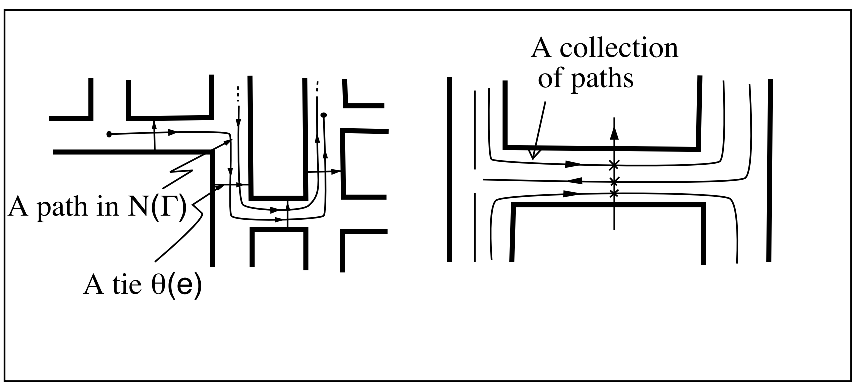


Figure 2: Transversal words

Proposition 1.2 *Let $\{\pi_i\}_{i=1\cdots k}$ be a set of disjointly embedded paths in $N(\Gamma)$. Let Γ' be the graph which is obtained by subdividing (if necessary) Γ at the points of $\partial(r \circ \pi_i)$ ($i = 1, \cdots, k$). Then the collection of longitudinal words $w(\pi_i)$ ($i = 1, \cdots, k$), together with the collection of transversal words $R_\pi^\perp(e)$, $e \in E(\Gamma')$, determines the embedding of the collection π, up to isotopy transverse to the ties and which fixes the extrem points pointwise.*

Proof: The longitudinal words define the collection and the order of the rectangles of $N(\Gamma')$ which are intersected by the paths. The transversal words give the respective position of the embedded paths in each rectangle. If $\cdots ee' \cdots$ is a subword occuring in the longitudinal word $w(\pi_i)$, we just have to connect the embedded segments $\pi_i \cap R(e)$ and $\pi_i \cap R(e')$ by an arc embedded transversally to the ties in the polygon $P(v)$ where $v = t(e) = i(e')$. Since the paths π_i are disjointly embedded, these arcs can be chosen to be disjointly embedded. $\diamondsuit$

1.2 Efficient Representatives

In what follows, all surfaces S have boundary and/or punctures and thus the fundamental group $\pi_1(S)$ is a free group.
We denote by $Homeo^+(S)$ the group of orientation preserving homeomorphisms of the surface S and $MCG(S)$ the *mapping-class group of* S, i.e. the group of isotopy classes of $Homeo^+(S)$ (see[Bi]).
Any homeomorphism f in a class $[g] \in MCG(S)$ induces a class $[f_\#]$ of outer automorphisms of $\pi_1(S)$ in the usual way.

Definition 1.3 ([BH1], [Lo2])

1. A *topological representative* (ψ, Γ) of $[f]$ in $MCG(S)$ consists of:
   - a graph Γ embedded in S with an identification of $\pi_1(\Gamma)$ with $\pi_1(S)$
   - a continuous map $\psi : \Gamma \to \Gamma$ so that
     (a) $\psi(V(\Gamma)) \subset V(\Gamma)$,
     (b) The induced map $\psi_\# : \pi_1(\Gamma) \to \pi_1(\Gamma)$, which is defined up to inner automorphism, belongs to the outer automorphism class $[f_\#]$,
     (c) For all e in $E(\Gamma)$, $\psi(e)$ is a reduced edge path,
     (d) The map ψ is induced by an embedding $\varphi : N(\Gamma) \to N(\Gamma)$ so that $\varphi(\Gamma)$ satisfies for all $e_i \in E(\Gamma)$, $r \circ \varphi(e_i) = \psi(e_i)$, where r is the retraction from $N(\Gamma)$ onto Γ defined above.

2. An *efficient representative* (ψ, Γ) of $[f]$ in $MCG(S)$ is a topological representative of $[f]$ such that for all positive integer k and for all edges $e \in E(\Gamma)$, $\psi^k(e)$ is a reduced edge path.

The reader may refer to figure 18 for an example of an efficient representative.

Remark 1.4
- We have used here the abuse of notation which identifies the abstract edge $e_i \in E(\Gamma)$ with the segment $\Gamma \cap R(e_i) \subset N(\Gamma)$. In what follows, we shall not distinguish between the graph and the embedded graph.
- Notice that $\pi_\varphi = \overline{\varphi(\Gamma) \setminus \varphi(V(\Gamma))}$ is a set of disjointly embedded paths in $N(\Gamma)$. This collection of paths is by proposition 1.2 uniquely determined by the longitudinal words $\varphi(e)$ and the transversal words $R^\perp_{\pi_\varphi}(e)$ ($e \in E(\Gamma)$). Therefore, the graph $\varphi(\Gamma)$ is also uniquely determined by these words.

Let us recall that the elements of the mapping class groups are classified, by the Nielsen-Thurston theorem ([Th1], [FLP]), into three classes : *finite order, pseudo-Anosov* and *reducible*. In this paper we won't use explicitly the properties of these differents classes and we refer the interested reader to the above mentioned papers.

Theorem 1.5 *([BH1],[Lo1],[FM])*
Let S be an oriented surface with boundary and/or punctures. Each irreducible isotopy class in $MCG(S)$ admits an efficient representative.

The proof of this theorem is constructive. For a homeomorphism f given by any automorphism induced on $\pi_1(S)$, there is a finite algorithm which gives an efficient representative if the class is irreducible. Otherwise, the algorithm enables one to find the reduction curves (for more details see [Ke1]). In [Lo1] and [FM] this theorem is proved in the case of the punctured disc. A similar algorithm also appears for free group automorphisms in [BH2] and [Lo3].
The *subdivided representative* (ψ_s, Γ_s) of (ψ, Γ) is the topological representative such that the graph Γ_s is the graph Γ subdivided at each pre-image under ψ of the vertices of Γ. The map ψ_s is obtained by rewriting ψ on the new graph Γ_s.
We will denote by D_N the disc with N punctures or equivalently N marked points. We will not distinguish beetween these two objects. In particular, the homotopies of D_N and the homotopies of D^2 relative to the N marked points are the same. We identify D_N with the unit disc of the complex plane, where the punctures lie on the real axis and are labelled according to their position along the axis. We denote $D_N^+ = \{z \in D_N | Im(z) \geq 0\}$.

Definition 1.6 ([Lo1], [Lo2])
A graph Γ embedded in D_N is *canonical* if it satisfies the following properties:

1. Γ has N edges $B(\Gamma) = \{b_1, \cdots, b_N\}$ such that $i(b_j) = t(b_j)$ for all $j = 1, \cdots, N$. Each edge of $B(\Gamma)$ is a closed curve which bounds a 1-punctured disc in D_N. These edges are called *boundary edges.*

2. $\Gamma \setminus B(\Gamma)$ is a tree.

3. All the vertices of $S(\Gamma) = \{v \in V(\Gamma) | \exists\, e_j \in B(\Gamma) \text{ s.t. } v = i(e_j)\}$ have valence three. They are called the *boundary vertices* and are labelled according to the corresponding puncture.

4. All the edges of $\Gamma \setminus B(\Gamma)$ are embedded in D_N^+.

Let (ψ, Γ) be an efficient representative of a pseudo-Anosov class $[f]$ in the punctured disc. It is called *canonical* if the graph Γ is canonical and the boundary edges are permuted under ψ.
In fact, the condition 4/ of 1.6 is not essential in our approach. It simplifies the combinatorics of some arguments in the second section. It is simply a matter of changing the embedding of the graph in the punctured disc by a conjugacy. The existence, up to conjugacy, of a canonical efficient representative for pseudo-Anosov isotopy classes of the punctured disc is proved in [Lo1].

1.3 Braids and Normal Dissection

1.3.1 Braids

Let $D^2 \times [0, 1]$ be the solid cylinder standardly embedded in $\mathbf{R}^3$ with basis $(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ (see figure 3).

Definition 1.7 1. A *geometric N-strand braid* β is a set of N paths $b_i(t)$, $t \in [0,1]$, disjointly embedded in $D^2 \times [0,1]$ such that:

(a) $b_i(0) \in D^2 \times \{0\}$ and $b_i(1) \in D^2 \times \{1\}$ for every $i \in \{1, \cdots, N\}$.

(b) The *strands* $b_i(t)$ are transverse to the discs $D^2 \times \{\theta\}$ for all t and $\theta \in [0,1]$.

(c) The set of points $b_i(1)$ (the *initial points*) (resp. $b_i(0)$ the *terminal points*) lie on an oriented arc properly embedded in $D^2 \times \{1\}$ (resp. $D^2 \times \{0\}$) called the *initial axis* (resp. *terminal axis*).

2. Two geometric braids β and β' are *equivalent* if and only if they are isotopic by an ambient isotopy ϕ_μ ($\mu \in [0,1]$) in the solid cylinder preserving the order of the initial and terminal points on their respective axis and such that for all μ, $\phi_\mu(\beta)$ satisfies the properties 1/, 2/ and 3/.

An equivalence class is called a *braid.*

The set of all equivalence classes of braids is the classical braid group $\mathcal{B}_N$ (see [Ar], [Bi]). A *representative* of a braid is a *regular projection* (without triple points) of a geometric braid, parallel to one of the directions, in a rectangle $[0,1] \times [0,1]$ (see figure 3). More precisely, the rectangle is embedded for instance in the yz-plane and both axis (initial and terminal) of the braid are isotoped to be parallel (in this case) to the y-direction. At each double point of the projection, the crossing-over strand is isotoped slightly off the rectangle in the positive half-space. We also assume that, at a given level $[0,1] \times \{t\}$, there is at most one crossing. All braids can be given by such representatives. The strands of a braid are labelled according to their position along the initial axis.
The *permutation* π_β *induced by a braid* $\beta \in \mathcal{B}_N$ is a permutation on the set $\{1, \cdots, N\}$, labelling the end points of the braid strands, defined as follows:
$\pi_\beta(i) = j$ if the terminal point of the strand i is the point j.
A braid has k *components* if its induced permutation has k cycles.

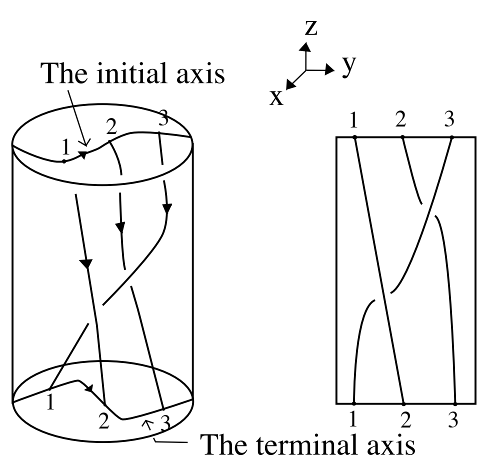


Figure 3: A braid and its y-projection

Definition 1.8 ([BGN])
Let β be a braid in $\mathcal{B}_N$ given by one of its representatives.

- The *crossing-letter* $s_{i,j}$ (resp. $s_{i,j}^{-1}$) represents the crossing of the strand i over the strand j from left to right (resp. right to left).

- A representative of β, obtained by reading the sequence of crossings in $[0,1]\times[0,1]$ as $[0,1]\times\{\theta\}$ varies from $[0,1]\times\{1\}$ to $[0,1]\times\{0\}$, defines a unique *crossing-word*, i.e. a finite word composed of letters $s_{i,j}^{\pm 1}$. Conversally, a crossing-word defines a unique representative of a braid, if any.

- A *block* for a representative of β is a set B of consecutive strands (i.e. strands whose initial points are consecutive) such that each strand not in B crosses all the strands in B in the same way (or does not cross them at all). A braid admits a *block partition* if it has a representative with at least a block non-reduced to a single strand.

- A braid with an even number of strands admits a *2-strand block partition* if it has a representative such that the strands (b_{2i-1}, b_{2i}) form a block B_i for all $i = 1, \cdots, N/2$.

These crossing-words are associated to a given representative of a braid. The crossing-words corresponding to all the representatives of a given braid are obtained by some obvious equivalence relations in the crossing letters $s_{i,j}$ coming from the braid relations (see [BGN]). If a braid representative admits a block partition, then one can decompose its crossing-word in *block letters* (see [BGN] for details).
We complete this section by a classical result about the relationship beetween the braid groups and the mapping class group $MCG(D_N)$:

Theorem 1.9 *([Ar],[Bi])*
The mapping-class group $MCG(D_N)$ of the punctured disc D_N is isomorphic to the quotient $\mathcal{B}_N/\mathcal{Z}_N$, where $\mathcal{Z}_N$ is the center of $\mathcal{B}_N$.

The center of $\mathcal{B}_N$ is known to be the cyclic subgroup of $\mathcal{B}_N$ generated by a single braid Δ_N, which corresponds to a Dehn twist H_Δ along the outer boundary ∂D^2 of D_N.

1.3.2 Normal Dissection

The normal dissection will be our essential tool for constructing braids. It provides a connection beetween dimension 2 and dimension 3. The goal is to transform an outer automorphism of $\pi_1(D_N)$, induced by an homeomorphism F of D_N, to a braid.
We identify the disc D^2 with the rectangle $A = [0,1]\times[-1,1]$ and the punctured disc D_N with a rectangle A_N with N marked points $\{p_1, \cdots, p_N\}$.

Definition 1.10 A *normal dissection* $\mathcal{S}$ in A_N is a set of $N+1$ oriented paths $\{\pi_1, \cdots, \pi_N; \mathcal{R}\}$ such that:

- The *arcs* π_i are disjointly embedded. Their initial points belong to the segment $[0,1]\times\{1\}$ of the boundary and their terminal points are the marked points p_i ($i = 1, \cdots, N$).

- The *axis* $\mathcal{R}$ is a properly embedded path in the rectangle and goes through all the marked points $\{p_1, \cdots, p_N\}$ of A_N.

By an isotopy of $\mathcal{R}$ in A_N fixing ∂A pointwise, we can always assume that the N marked points p_i lie on the axis $[0,1]\times\{0\}$.
Let A, B be two points in an arc π_i, we denote by $\overset{\frown}{AB}$ the arc on π_i bounded by the points A and B.
If two points A and B belong to the axis $\mathcal{R}$, $\overline{AB}$ will design the segment of $\mathcal{R}$ which lies beetween these points. Such a segment is *elementary* if it contains no marked point in its interior.

A *trivial disc* in a normal dissection is a disc whose boundary is $\overset{\frown}{AB} \cup \overline{AB}$, where $A, B \in \pi_i \cap \mathcal{R}$ $(i = 1, \cdots, N)$ and such that $\overline{AB}$ is elementary (see figure 4).
A *reduced normal dissection* is a normal dissection without trivial discs.

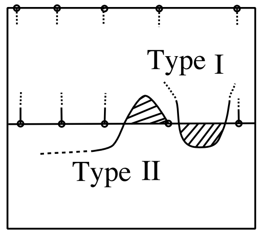


Figure 4: Trivial discs

The relationship beetween braids and normal dissections can be understood in both directions. The construction of a normal dissection from a braid is given in the book [BZ]. The inverse operation, i.e. how to construct a braid from a normal dissection, is due to U.Keil ([Ke2]) and has never been published. So, for completeness, we give here a short proof which relies on lemma 1.12.

Theorem 1.11 *([BZ],[Ke2])*

- *A braid $\beta \in \mathcal{B}_N$ defines a unique isotopy class of normal dissection $\mathcal{S}(\beta)$ in A_N, where the isotopy fixes the set of marked points $\{p_1, \cdots, p_N\}$ and the boundary ∂A_N pointwise.*

  In such an isotopy class, there exists a unique reduced normal dissection $\mathcal{S}_r(\beta)$.

- *From a reduced normal dissection $\mathcal{S}$, there exists a finite algorithm which enables us to reconstruct a representative of a braid $\beta \in \mathcal{B}_N$ so that $\mathcal{S}_r(\beta) = \mathcal{S}$.*

The reconstruction algorithm is presented below (see figures 5 and 6).
We denote by S_j the initial point in $[0,1] \times \{1\}$ of the arc π_j in $\mathcal{S}$.
Let F_i $(1 \leq i \leq N)$ be the first intersection point of the arc π_i with the axis $\mathcal{R}$. A *band* Bl in $\mathcal{S}$ is a maximal set of consecutive arcs $\{\pi_i, \pi_{i+1}, \cdots, \pi_{i+k}\}$ such that the segments $\overline{F_j F_{j+1}}$ are elementary for all j in $\{i, \cdots, i+k-1\}$.
Let π be an arc in a band Bl. A point $P \in \pi \cap \mathcal{R}$ is called a *return point* for Bl if one of the segments $\overline{PF_i}$ or $\overline{F_{i+k}P}$ does not contain any point of $(\cup \pi_i) \cap \mathcal{R}$. The main lemma of the reconstruction algorithm of Keil ([Ke2]) states that, for a non trivial reduced normal dissection, such a return point always exists.

Lemma 1.12 *Let $\mathcal{S}$ be a non trivial reduced normal dissection. There exists a return point P in $\mathcal{S}$.*

<u>Proof:</u> We use the notations introduced above. Observe that, by definition, any arc of a normal dissection belongs to a band, possibly reduced to this single arc. One wants to prove that there exists a band in $\mathcal{S}$ which has a return point. One considers the band Bl containing the first arc π_1. Let $\{\pi_1, \cdots, \pi_k\}$ the arcs in this band Bl. If there is a return point for this band, then we are done. Assume that there is no such return point. Let Q be the intersection point of the set of arcs π_i, $i = 1, \cdots, N$, in $\mathcal{S}$ with the axis $\mathcal{R}$, such that Q follows F_k along $\mathcal{R}$ and $\overline{F_k Q}$ is elementary. By assumption, Q does not belong to an arc of the band Bl. Let π_{k+l} be the arc of $\mathcal{S}$ containing Q $(l > 0)$. One considers the sub-arc of π_{k+l}, denoted by a, which connects the initial point S_{k+l} to Q. One connects the point Q to a point M in the upper part of the boundary of A_N, $[0,1]\times\{1\}$, by an arc b parallel to the segment in π_k connecting S_k to F_k. The union of the arc $a\cup b$ with the interval in $[0,1]\times\{1\}$ connecting M to S_{k+l} bounds a punctured disc embedded in A_N. All the arcs $\pi_{k+1},\cdots,\pi_{k+l-1}$ are embedded in this disc and their first intersection point with the axis $\mathcal{R}$ lie in the interval $\overline{QF_{k+l}}$. One now considers the normal dissection in A_l formed by these arcs, the arc π_{k+l} if its terminal point belongs to $\overline{QF_{k+l}}$ and the axis $\mathcal{R}$. This new normal dissection has a number of arcs strictly lesser than the one of $\mathcal{S}$ and all the first intersection points of its arcs (that is $F_{k+1},\cdots,F_{k+l-1}$ and possibly F_{k+l}) lie in an interval strictly contained in $\mathcal{R}$. Moreover, if a band of this normal dissection has a return point, then the corresponding band in the initial normal dissection A_N has a return point. One now consider the band Bl_1 containing the first arc of this new normal dissection, this is the arc π_{k+1} of $\mathcal{S}$. One can now repeat the argument above. An easy induction allows to conclude, because at each step,

- either one finds a band with a return-point,
- or one constructs a new normal dissection, contained in the preceding, which has a number of arcs strictly lesser than the preceding one, and such that all the first intersection points of its arcs lie in an interval stricly contained in the preceding (and all are strictly contained in $\overline{QF_{k+l}}$).

Lemma 1.12 is proved. $\diamondsuit$

Suppose that a return point P belongs to the arc π of the band Bl. Let $Q\in[0,1]\times\{1\}$ be such that the segment PQ is vertical. Observe that, by definition of the return point P, this segment can be assumed not to intersect any of the arcs π_j of the normal dissection. We call *reconstruction disc* $\mathbf{D}$ (see figure 5) the embedded disc whose boundary is $\widehat{S_jP}\cup PQ\cup QS_j$. This disc is such that:

- It contains the arcs in the band Bl intersecting the axis $\mathcal{R}$ beetween the first intersection of π with $\mathcal{R}$ and the return point.
- It contains in its boundary the part of the arc π going from its initial point to the return point P.
- It does not intersect any other arcs of the normal dissection.

We identify the cylinder $D^2\times[0,1]$ with the cube $A\times[0,1]$ standardly embedded in $\mathbf{R^3}$. Let us now describe the reconstruction algorithm.

$\diamondsuit$ *Data of the reconstruction algorithm:*

- $\mathcal{S}_0$ a reduced non trivial normal dissection in $A_N\times\{1\}$.
- β_0 the x-projection of the trivial braid, whose terminal points are the initial points of $\mathcal{S}_0$ and the initial points belong to $A\times\{0\}$.

The operations (see figure 6)
$\diamondsuit$ We choose a return point P for some π_j in a band $Bl=\{\pi_i,\cdots,\pi_j,\cdots,\pi_{i+k}\}$ $(1\leq i\leq N,\ 0\leq k\leq N-i,\ i\leq j\leq i+k)$.

$\diamondsuit$ We consider the corresponding reconstruction disc $\mathbf{D}$. We make a copy $\mathbf{D}'$ of this disc in $A \times \{1-\epsilon\}$ for some small $\epsilon > 0$. We shall denote by x' a point of $\mathbf{D}'$ which is the copy of $x \in \mathbf{D}$. The two curves $\partial\mathbf{D}$ and $\partial\mathbf{D}'$ bound an annulus in $A \times [1-\epsilon, 1]$.
$\diamondsuit$ We first apply an isotopy supported by the above annulus which transforms the arc $\overset{\frown}{S_jP} \subset \partial\mathbf{D}$ to $\overset{\frown}{S'_jP'} \subset \partial\mathbf{D}'$. Then we make an isotopy in $\mathbf{D}'$ which transforms $\overset{\frown}{S'_jP'}$ to $\overset{\frown}{S'_jQ'}$ in $\mathbf{D}'$ and we connect $\overset{\frown}{S'_jQ'}$ to the point $P \in \mathbf{D}$ by the segment $Q'QP$.
$\diamondsuit$ We now define a new dissection $\mathcal{S}'_1$ in $A \times \{1\}$. We replace the path π_j of $\mathcal{S}_0$ which is written as $\pi_j = \overset{\frown}{S_jP}\ \tilde{\pi}_j$ by the path $\pi^1_j = \overset{\frown}{QP}\ \tilde{\pi}_j$ and we keep the other paths π_i of $\mathcal{S}_0$ unchanged.

$\diamondsuit$ The new arc $\overset{\frown}{S'_jQ'}$ is the projection along the x-direction (see figure 5) of the arc $\overset{\frown}{S'_jP'}$. It defines a representative of a braid β_1 in the cylinder $A \times [0, 1]$. This braid is so that the strand which, in β_0, ended at S_j, has now the point Q as terminal point. It crosses over the strands whose terminal points were contained in the interior of the segment S_jQ.

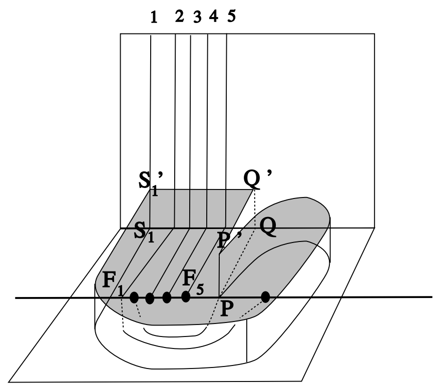


Figure 5: A reconstruction disc

The crossing-word corresponding to β_1 has the form $s^{-1}_{j,j-1}s^{-1}_{j,j-2}\cdots s^{-1}_{j,i}$ (resp. $s_{j,j+1}s_{j,j+2}\cdots s_{j,j+k}$) if P is on the left (resp. on the right) of Bl.
$\diamondsuit$ The braid β_0 has now been replaced by a braid β_1 and the normal dissection $\mathcal{S}_0$ by $\mathcal{S}'_1$. This dissection is not necessarely reduced. By an isotopy in $A \times \{1\}$, we obtain a reduced dissection $\mathcal{S}_1$.
$\diamondsuit$ We iterate this process until the dissection $\mathcal{S}_i$ is trivial. At each step, the crossing-word obtained is placed at the right of the preceding crossing-word. In order to obtain the final crossing-word, we have to relabel the strands at each step of the reconstruction.
This process obviously stops since, at each step, the number of intersection points of the arcs π_i of the normal dissection with the axis $\mathcal{R}$ is strictly decreasing.
A normal dissection is given combinatorially by the finite set of intersection points $\bigcup_i(\pi_i \cap \mathcal{R})$. These intersection points are ordered along both the axis $\mathcal{R}$ and each path π_i. Such a combinatorial dissection uniquely defines a normal dissection.

In our construction of a braided branched surface, the fact for a braid to have a 2-strand block partition will play an important role.
Our next lemma gives a sufficient condition to decide if a given normal dissection defines a braid γ with a 2-strand block partition.

Lemma 1.13 *Let $\mathcal{S}$ be a reduced normal dissection in A_N (N is even) and let $\gamma \in \mathcal{B}_N$ be the associated braid.*
A sufficient condition on $\mathcal{S}$ for γ to admit a 2-strand block-partition is that

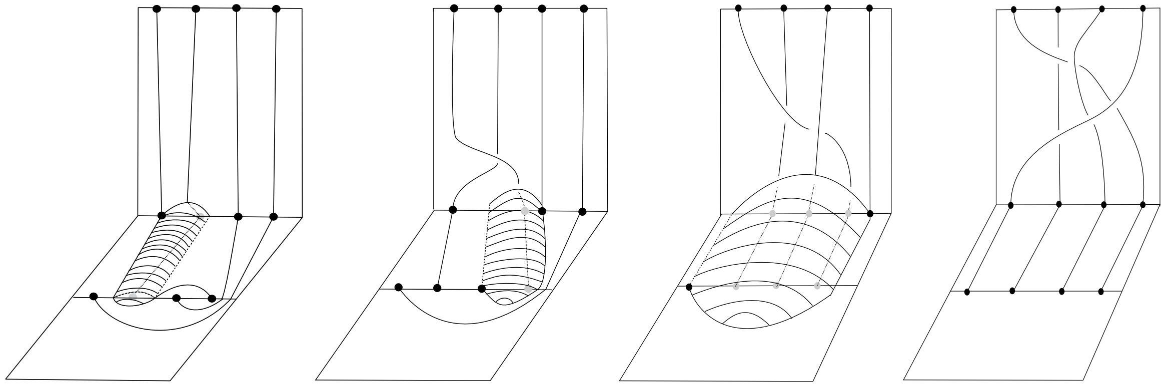


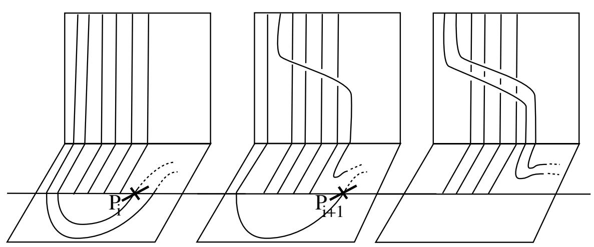
Figure 6: Example of a reconstruction

- *The number of intersection points of any two arcs π_{2k-1}, π_{2k} of the normal dissection with the axis $\mathcal{R}$ differs at most by one. The terminal points of two such arcs are consecutive along $\mathcal{R}$.*
- *The i^{th} intersection point of the arc π_{2k-1} with the axis $\mathcal{R}$, according to the orientation of π_{2k-1}, is consecutive along $\mathcal{R}$ with the i^{th} intersection point of π_{2k} if it exists.*

Moreover, if these conditions are satisfied, the reconstruction algorithm can be applied so that the resulting braid representative admits a 2-strand block partition.
See figure 7.

Proof: The proof of this lemma relies on the following fact:

If the conditions of the lemma are satisfied then
If an arc π_{2k-1} (resp. π_{2k}) contains a return point P_i at the i^{th} step of the reconstruction algorithm, i.e. in $\mathcal{S}_{i-1}$, then the arc corresponding to the arc π_{2k} (resp. π_{2k-1}) in the new normal dissection $\mathcal{S}_i$ contains a return point P_{i+1}.



S_{i-1} $\qquad$ S_i

Figure 7: Reconstruction of a 2-strand block

The proof of this property is clear (see figure 7).
This property implies that the reconstruction algorithm can be applied as follows:
If an arc $\pi_{2k-1}^{(i-1)}$ (resp. $\pi_{2k}^{(i-1)}$) of the normal dissection $\mathcal{S}_{i-1}$ is modified at the step i then we modify the arc $\pi_{2k}^{(i)}$ (resp. $\pi_{2k-1}^{(i)}$) of $\mathcal{S}_i$ at the step $(i+1)$.
If this rule is respected at each step, then the two strands labelled $2k-1$ and $2k$ remain parallel at each step and for every k. One then checks that the resulting representative of the braid β admits a 2-strand block partition. $\diamondsuit$

Normal dissection and suspension

Let F be in $Homeo^+(D^2)$ and $P = \{x_1, \cdots, x_N\}$ be a periodic orbit for F. We choose an axis $\mathcal{R}$ embedded in D^2 passing through the points of P.
A *suspension flow* $(F_t)_{t\in\mathbf{R}}$ *of* F is a flow in the solid torus which we consider to be $D^2\times[0,1]$ together with the identification of $D^2 \times \{0\}$ with $D^2 \times \{1\}$ by the identity map. This flow is transverse to the fibers $D^2 \times \{\theta\}$, $\theta \in S^1$. Its return-homeomorphism on $D^2 \times \{1\}$ is F, possibly composed with a power of H_Δ (the Dehn-twist along ∂D^2). Up to isotopy of the solid torus which fixes the boundary pointwise, such a suspension flow $(F_t)_{t\in\mathbf{R}}$ is not uniquely defined. It is uniquely defined only when we have fixed the twisting of its orbits on the boundary of the solid cylinder $D^2 \times [0,1]$.

Let $(F_t)_{t\in\mathbf{R}}$ be a fixed suspension flow of F. The suspension under this flow of the periodic P of F is the periodic orbit $\{F_t(x_i)\ ;\ t \in \mathbf{R},\ x_i \in P\}$ of $(F_t)_{t\in\mathbf{R}}$. It is a closed braid $\overline{\beta}$ in the solid torus, where $\beta \in \mathcal{B}_N$ is well defined up to conjugacy. The choice of a braid in this conjugacy class is done by choosing an axis $\mathcal{R}$ as above.
The suspension of a periodic orbit P for two distinct suspension flows of F defines two conjugacy classes $[\beta]$ and $[\beta']$ in $\mathcal{B}_N$ where β and β' differ by an element of the center $\mathcal{Z}_N$. In other words, the suspension of a periodic orbit P is well defined up to conjugacy in $\mathcal{B}_N/\mathcal{Z}_N$.

If β is a braid in $\mathcal{B}_N$, we denote by M^3_β the solid torus with a distinguished closed braid $\overline{\beta}$ or the complement of this closed braid in the solid torus. The equivalence beetween these two objects is the same as the equivalence beetween the disc D_N with N punctures or N marked points.
Let β be a braid in $\mathcal{B}_N$ and $[f_\beta] \in MCG(D_N)$ be the induced isotopy class. We recall that the N marked points are a finite collection P of periodic orbits for every F in $[f_\beta]$.
A *suspension flow* $(F^\beta_t)_{t\in\mathbf{R}}$ *of* $F \in [f_\beta]$ *associated to* $\beta \in \mathcal{B}_N$ is a suspension flow of F which has $\overline{\beta}$ (and not $\overline{\beta\Delta^k_N}$) as the suspension of the periodic orbits P. This flow is defined on the manifold M^3_β (and not $M^3_{\beta\Delta^k_N}$).
The following lemma gives the connection beetween normal dissection in A_N and the periodic orbits of period N for a suspension flow of a given disc homeomorphism.

Lemma 1.14 *Let F be an orientation preserving homeomorphism of the disc D^2, fixing ∂D^2 pointwise, defined up to a power of H_Δ. Let $P = \{x_1, \cdots, x_N\}$ be a periodic orbit of F. Let $\mathcal{R}$ be an arc properly embedded in D^2 so that $\mathcal{R}$ passes through all the points of P. We denote by D^+ one of the components of $D^2\backslash\mathcal{R}$. If B_P is a collection of disjointly embedded arcs in D^+ which connect $\partial D^2 \cap D^+$ with the points of P, then $\mathcal{C}_{Per} = \{F(B_P); \mathcal{R}\}$ is a normal dissection. The closure of the corresponding braid is ambient isotopic in $D^2 \times S^1$ to a suspension of the periodic orbit P. The choice of a power of H_Δ is equivalent to fix a suspension flow.*

<u>Proof:</u> The paths in $F(B_P)$ are disjointly embedded since F is a homeomorphism. The arc A has been chosen to satisfy the properties of an axis with respect to the set P of marked points. Therefore, $\mathcal{C}_{Per}$ is a normal dissection. Then, we get a braid α in $\mathcal{B}_N$.
It defines also an automorphism $F_\#$ induced by F on $\pi_1(D_N)$ whose generators have been identified to the set of paths B_P connected to a base-point in ∂D^+.
A suspension $\{F_t(x)\ ;\ x \in P,\ t \in \mathbf{R}\}$ of a periodic orbit P of F defines a closed braid $\overline{\beta}$. One fixes the suspension flow so that its return homeomorphism is F and not $F \circ H^k_\Delta$. The braid β induces an automorphism ψ on $\pi_1(D_N)$, whose generators have been identified

with the set of paths B_P connected to the same base-point as above. By theorem 1.9, we have $F_{\#} = \psi$ since F is the return homeomorphism of the suspension flow $(F_t)_{t\in\mathbf{R}}$ on D_N. We obtain that $\alpha = \beta$. Some choices have been done in this construction:

- Choice of a base-point, which implies the choice of D^+.
- Choice of the axis $\mathcal{R}$, which implies in particular the ordering of the points along $\mathcal{R}$. This also implies the choice of the generators for $\pi_1(D_N)$.
- Choice of the power of H_Δ.

Changing the two first choices induces a change of the braid by a conjugacy in $\mathcal{B}_N$. This does not change the closed braid up to ambient isotopy in $D^2 \times S^1$. Changing F by a power of H_Δ is equivalent to compose the braid by an element of the center $\mathcal{Z}_N$. This fixes the choice of the suspension flow in $D^2 \times [0,1]$. $\diamondsuit$

2 Construction of a *supporting braid*

Our goal is to construct a special braided branched surface $W_{\overline{\beta}}$ as announced in the main theorem. To this end, we assume that an efficient representative (ψ, Γ) of $[f_\beta] \in MCG(D_N)$ for the pseudo-Anosov braid $\beta \in \mathcal{B}_N$ is given. Up to conjugacy (in $\mathcal{B}_N$ or in $MCG(D_N)$), we assume that (ψ, Γ) is canonical (see section 1.2).
The branched surfaces we consider in this paper are defined in section 3 (definitions 3.2 and 3.3). For the strategy of our construction, let us recall from the introduction that they are embedded in the solid torus $D^2 \times S^1$ so that the branched locus is a collection of intervals disjointly embedded in a single meridian disc, say $D^2 \times \{0\} \simeq D^2 \times \{1\}$. The complement of the branch locus is a collection of rectangles which are embedded in $D^2 \times (0,1)$ transversally to the meridian discs. As a result, for this class of branched surfaces, the boundary of the rectangles in $D^2 \times [\epsilon, 1-\epsilon]$ (for some small $\epsilon > 0$) is a braid with a 2-strand block partition, which we call the *supporting braid.*
The goal of this section is construct effectively this supporting braid so that the block partition is explicit.
In the next section, we will make explicit the identification of the branched locus in order to obtain the embedding of the branched surface in M^3_{β}.

2.1 Preliminaries

In this subsection, we define a collection of marked points from the efficient representative (ψ, Γ). The idea is to make all these points lie on a particular axis so that their ordering along the axis satisfy some properties with respect to their position on the graphs Γ and $\varphi(\Gamma)$.
Let us recall that, for a given graph Γ embedded in a surface (here, it is the punctured disc), we have defined the fibered neighborhood $N(\Gamma)$ as a union of rectangles and polygons embedded in the surface (see section 1.1). The intersection of the boundaries of two rectangles contains at most two points in $\partial N(\Gamma)$. These points are called *corners.*
The punctured disc D_N is identified with the unit disc in the complex plane, with all the punctures lying on the x-axis. We orient D_N so that ∂D^2 is oriented counterclockwise. Let the orientation of the x-axis of D_N be the one induced by D^+_N.
In what follows, we shall assume that a canonical efficient representative (ψ, Γ) is given as well as the embedding $\varphi : N(\Gamma) \to N(\Gamma)$ (see definition 1.3). Let us recall that $\varphi : N(\Gamma) \to N(\Gamma)$ is defined as the embedding so that $r \circ \varphi : \Gamma \to \Gamma$ is the map $\psi : \Gamma \to \Gamma$,

where r denotes the retraction of $N(\Gamma)$ onto Γ (see section 1.1). We shall often use (ψ, Γ) as well as the subdivided representative (ψ_s, Γ_s). We recall that the transversal words for $\varphi(\Gamma)$ (or $\varphi_s(\Gamma_s)$) at an edge e are denoted by $R^{\perp}_{\pi_\varphi}(e)$ (or $R^{\perp}_{\pi_{\varphi_s}}(e)$)
Since (ψ, Γ) is efficient, we can extend the embedding $\varphi : N(\Gamma) \to N(\Gamma)$ to a homeomorphism $F : D_N \to D_N$ which is pseudo-Anosov (see [BH1] or [Lo2] for details).
A priori, the homeomorphism F induces a non trivial rotation on the boundary ∂D_N. We extend F to a homeomorphism $\widetilde{F} : \widetilde{D}_N \to \widetilde{D}_N$ by assuming that $D_N \hookrightarrow \widetilde{D}_N$ and $\overline{\widetilde{D}_N \setminus D_N}$ is a topological annulus A so that $\widetilde{F}_{|A}$ is a twist map which satisfy $\widetilde{F}_{|\partial D_N} = F_{|\partial D_N}$ and $\widetilde{F}_{|\partial \widetilde{D}_N} = Id_{\partial \widetilde{D}_N}$. By a slight abuse of terminology, we will ommit the tilde and speak of the *pseudo-Anosov homeomorphism F on D_N*.

Remark 2.1 The homeomorphism F is not uniquely defined by the above conditions. Two such homeomorphisms differ by a power of a Dehn twist H_Δ. In what follows, we will suppose that such a homeomorphism has been fixed.

2.1.1 Accessibility

Definition 2.2
- A point x in $\partial N(\Gamma)$ is *accessible* if there exists a path embedded in $\overline{D^+_N \setminus N(\Gamma)}$ whose extremities are x and a point of $\partial D^2 \cap D^+_N$.
- A point y in $N(\Gamma)$ is *exterior*, with respect to the embedding $\varphi : N(\Gamma) \to N(\Gamma)$, if one of the segments between y and $\partial N(\Gamma)$ on the tie containing y does not intersect $\varphi(\Gamma)$.

Let us recall that a canonical graph Γ has a subset $B(\Gamma) \subset E(\Gamma)$ whose elements are the boundary edges, which are loops bounding once punctured discs. We denote $\mathcal{T} = \Gamma \setminus B(\Gamma)$ the complementary tree. Let us denote by $N(\mathcal{T})$ the fibered neighborhood of this tree embedded in the disc D_N.
We have observed in section 1.2 that a canonical embedding can be obtained by a conjugacy. By the same argument, any point of $\partial N(\mathcal{T})$ can be made accessible maybe after another conjugacy (see figure 8). The notion of exterior point is very easy to check. Indeed, from the definition of the transversal words (see definition 1.1), in every rectangle $R(e)$, the transversal word $R^{\perp}_{\pi_{\varphi_s}}(e) = I^{\epsilon_1}_{i_1} \cdots I^{\epsilon_k}_{i_k}$ is such that all the points beetween $\varphi_s(e_{i_1})$ and $\partial N(\Gamma)$ or beetween $\varphi_s(e_{i_k})$ and $\partial N(\Gamma)$ are exterior (see figure 2).
Let us consider a vertex v_s of the subdivided representative (ψ_s, Γ_s) such that the image of a direct turn (e_i, e_j) at v_s under the embedding φ_s is exterior. The existence of such a vertex and of such a turn is clear from the previous observation. If we gather these observations, we obtain the following:

Proposition 2.3 *With the above notations, the embedding of the efficient representative (ψ, Γ) of $[f_\beta] \in MCG(D_N)$ can be realized, up to a conjugacy in $MCG(D_N)$, so that it is canonical and there exists a direct turn τ_s of Γ_s at a vertex $v_s \in V(\Gamma_s)$ satisfying:*

i) The points of the segment $\varphi_s(\tau_s) \cap P(\psi_s(v_s))$ are exterior.

ii) The corner corresponding to the turn τ_s in Γ_s is accessible.

The proof of this proposition is obvious from the above observations and the basic definitions of section 1.1. $\diamondsuit$
From now on, the embedding of the canonical efficient representative will be the one given by this proposition.

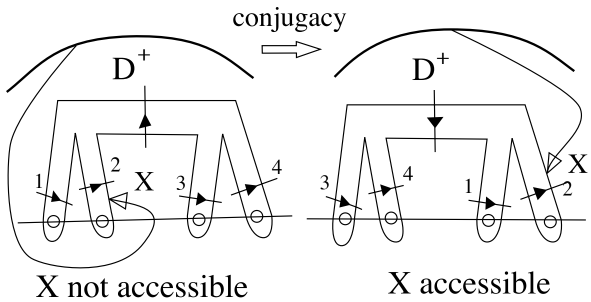


Figure 8: How to make a turn accessible

2.1.2 Nailed points and G-points

Let v_s be a vertex of Γ_s as in proposition 2.3. If v_s does not belong to $V(\Gamma)$, then we subdivide the graph Γ at v_s. The application ψ on Γ is rewritten in the new graph (see section 1.1). We obtain a new efficient representative. It will still be denoted by (ψ, Γ) in what follows.
We orient the tree $\mathcal{T}$ toward v_s. We blow up the accessible corner to an interval on $\partial N(\Gamma)$ that we call the *terminal side* (see figure 19).
Let us denote by $\mathcal{Y}_0$ the set of all intersection points $\varphi(\Gamma) \cap (\bigcup_{v \in V(\Gamma)} P_v - side)$, where the P_v-sides have been defined in section 1.1. For each boundary vertex v_{b_j}, there are two $P_{v_{b_j}}$-sides which intersect the boundary edge b_j. Let us denote by $\mathcal{Y}_1$ the subset of $\mathcal{Y}_0$ whose points belong to the above $P_{v_{b_j}}$-side for all $b_j \in B(\Gamma)$.

Definition 2.4 The set of *nailed points* is the set $\mathcal{Y} = \mathcal{Y}_0 - \mathcal{Y}_1$ in $\varphi(\Gamma) \subset N(\mathcal{T})$. The set of *G-points* is the set $\varphi^{-1}(\mathcal{Y})$ in $\mathcal{T} \subset N(\mathcal{T})$.

By definition, a G-point belongs to an edge e of Γ. The exact position of the G-points on an edge e is not essential for our purpose. What is important for us is their respective positions along the oriented edge e. So we will assume that all the G-points belong to some P_v-side, with $v \in V(\Gamma_s)$. From the above definition, the next proposition is obvious.

Proposition 2.5 *The number of nailed points and G-points is even and is equal to* $\mathcal{K} = 2[Card(E(\Gamma_s)) - Card(\psi^{-1}(B(\Gamma)))]$.

Our aim is to construct a normal dissection as defined in section 1.3.2. To this end, we will need, at some step of our construction, to detect and order the intersection points of the arcs of the normal dissection with its axis. To this end, we shall apply the two following lemmas:

Lemma 2.6 *Let c_1, c_2 be two paths going through a same polygon $P(v)$ (see figure 9). Then c_1 and c_2 have an essential intersection point in the polygon $P(v)$ if and only if the intersections points of a same path c_i $(i = 1, 2)$ with $\partial P(v)$ are not consecutive, among the points of $(c_1 \cup c_2) \cap \partial P(v)$, in the cyclic ordering at v. An intersection point is essential if it cannot be removed by an homotopy supported in the interior of the polygon.*

Lemma 2.7 *Let c_1, c_2 be two oriented paths disjointly embedded in $N(\Gamma)$ and going through a same polygon $P(v)$.*
Let $\mathbf{p}$ *be an other oriented path intersecting both c_1 and c_2 in an essential way (see lemma 2.6).*

The intersection point $c_1 \cap \mathbf{p}$ follows the intersection point $c_2 \cap \mathbf{p}$ along $\mathbf{p}$ if and only if the intersection points $c_2 \cap \partial P(v)$ are consecutive with the initial point of $\mathbf{p} \cap P(v)$ along $\partial P(v)$ oriented according to the cyclic order at v.

These two lemmas are easy to check (see figure 9). $\diamondsuit$

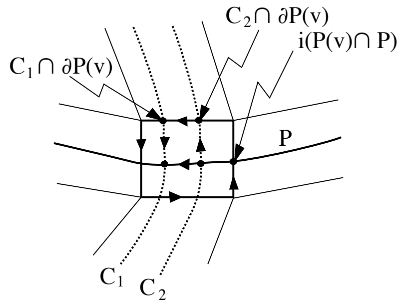


Figure 9: Intersections of paths in a polygon

Our first step will be to construct a collection of paths embedded in $N(\mathcal{T})$ whose image under φ and thus under F will be the arcs of our normal dissection.
The paths we are going to consider end at the G-points given in definition 2.4. Their initial points lie along the terminal side (see the definition after proposition 2.3). These paths define $\mathcal{K}$ longitudinal words (where $\mathcal{K}$ is given by proposition 2.5) in the graph Γ_G defined as the graph Γ subdivided at the G-points. The proposition 2.3 will enable us to connect the initial points of these paths to $\mathcal{K}$ points in ∂D^+ by $\mathcal{K}$ arcs disjointly embedded in $\overline{D^+ \setminus N(\Gamma)}$.

2.2 Construction of the supporting braid

Let us recall that the boundary of a polygon $P(v)$ is oriented according to the orientation induced by the one of D_N, i.e. counterclockwise. The P_v-sides are then oriented according to this orientation. Among these P_v-sides, we distinguish between the incoming one, denoted by $F_i(v)$, and the outgoing one, denoted by $F_o(v)$. For each vertex $v \in V(\Gamma)$, there is exactly one outgoing P_v-side $F_o(v)$. We consider the terminal side as the outgoing one at v_s (see the definition of v_s given after proposition 2.3).

2.2.1 Generating paths

Definition 2.8 A *set of generating paths* is a set $\pi = \{\pi_1, \cdots, \pi_{\mathcal{K}}\}$ of $\mathcal{K}$ oriented paths in $N(\mathcal{T})$ which satisfy:

1. Their initial (resp. terminal) points belong to the terminal side (resp. are the G-points). Their longitudinal words are the unique reduced words in $\mathcal{T}_G$ from v_s to the G-point they contain.

2. The generating paths are disjointly embedded in $N(\mathcal{T})$.

   Let $G_\pi(e)$ be the set of generating paths whose G-points belong to a same edge e of $\mathcal{T}$.

3. The paths in $G_\pi(e)$ intersect transversally the ties in a same component of $R(e) - e$. By convention, we assume that this component is the one containing the half-ties $\{x\} \times [0, 1]$ ($x \in e$) of the rectangle R(e) (see the parametrization of the rectangles in section 1.1).

4. The intersection points of the generating paths in $G_\pi(e)$ with the boundary $\partial P(v)$ ($v \in V(\mathcal{T})$) of any polygon $P(v)$ they pass through are consecutive along the union of the incoming P_v-side and along the outgoing P_v-side.

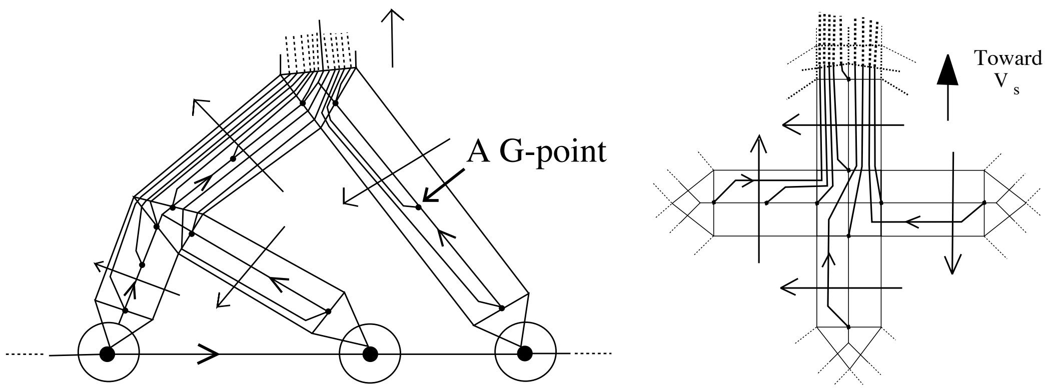


Figure 10: Some generating paths

Remark 2.9 There exists several possible sets of generating paths. The existence and non-uniqueness of the set of generating paths is clear from the definition. Observe that the rule 4/ in definition 2.8 implies that the intersection of the generating paths in $G_\pi(e)$ with any rectangle $R(e')$ ($e' \in E(\mathcal{T})$) they pass through is a collection of segments disjointly embedded in a same component of $R(e') - e'$.

In the following, we suppose that a set of generating paths has been chosen once and for all.

Labelling the generating paths and the G-points:
The generating paths are labelled according to the inverse ordering of their initial points along the terminal side. This defines an order on the set of generating paths, denoted by $\prec_{v_s}$.
The label of a G-point is the label of the generating path which contains it.
The definition of the generating paths has been given so that the above labelling satisfies the following property:

Proposition 2.10 *All the G-points which belong to a same edge e of $\mathcal{T}$ have consecutive indices. Moreover, they are ordered according to the orientation of the edge e. More precisely:*
If x_i, x_j are two G-points in e, then $i < j \Leftrightarrow x_j$ follows x_i along e.

Proof: Let v be in $V(\Gamma)$ and $e \in E(\mathcal{T})$ be such that $v = t(e)$. We denote by $G(v)$ the set of generating paths passing through $P(v)$ and by $G_e(v) \subset G(v)$ the subset containing the generating paths passing through the rectangle $R(e)$.
The following two properties are easy to check:

- The inverse ordering of the intersection points $\pi_j \cap F_o(v)$ ($\pi_j \in G(v)$) according to the orientation of $F_o(v)$, agrees with the ordering of the intersection points $\pi_j \cap (\bigcup F_i(v))$, according to the orientation of the incoming P_v-sides.
- The ordering of the intersection points $\pi_j \cap (\bigcup F_i(v))$ ($\pi_j \in G_e(v)$) according to the orientation of the incoming P_v-sides agrees with the inverse ordering of the intersection points $\pi_j \cap F_o(i(e))$, according to the orientation of the outgoing $P_{i(e)}$-side.

This comes from lemma 2.6 and the fact that the generating paths are disjointly embedded. The proposition 2.10 is then obtained from the rules 2/, 3/ and 4/ of definition 2.8. The rules 2/ and 4/ imply that the intersection points of the generating paths ending at a same edge $e \in E(\mathcal{T})$ with the corresponding incoming P_v-side $F_i(t(e))$ are consecutive along $F_i(t(e))$. The rule 3/ implies that the generating paths ending at a same edge e are ordered along the corresponding incoming side $F_i(t(e))$ according to their position along e. The two properties above imply that the respective positions of these paths are preserved up to the terminal side. $\diamondsuit$

Labelling the edges:
The edges of $\mathcal{T}$ are labelled according to the following rule:
Let e_i, e_j be in $E(\mathcal{T})$. Then $i < j$ holds if and only if the G-points in e_i have smaller indices than the G-points in e_j.

This labelling makes sense by proposition 2.10.
We call *boundary paths* the N reduced paths c_i in $\mathcal{T}$ going from the boundary vertices v_{b_i} $(i = 1, \cdots, N)$ to the vertex v_s (see definition 1.6 for the labelling of the boundary vertices). The following claim shows that the labelling of the edges is consistent with the labelling of the punctures.

Claim 1: According to the above labelling,
Let c_i, c_j be two boundary paths such that $i < j$. Then the indices of all the edges in c_i are less or equal than the indices of the edges in c_j. In particular, the first edge e_1 (resp. the last edge) belongs to the boundary path c_1 (resp. c_N).
This claim comes from definition 2.8 and the labelling of the generating paths.

2.2.2 The accordion axis

Proposition 2.11 *There exists an oriented arc A, properly embedded in D^+, whose extremities are the extremities of the axis of D^2 and which passes through all the G-points. The ordering along A of the G-points agrees with their labelling.*

Proof: For the construction of the curve A, we just have to find a collection of paths $p_{i,i+1}$ connecting the last G-point x_k in an edge e_i to the first G-point x_{k+1} in e_{i+1}, for all $i = 1, \cdots, card(E(\mathcal{T}) - 1)$. Then A will be the ordered union of all the intervals $e \cap R(e)$ $(e \in E(\mathcal{T}))$ with these paths $p_{i,i+1}$.
The G-points we want to connect have consecutive indices k, $k+1$. Therefore, by definition of the labelling, there exists a polygon $P(v)$ $(v \in V(\Gamma))$ such that the intersection points of the corresponding generating paths π_k, π_{k+1} with the union of the incoming P_v-sides are consecutive. We construct a path parallel to π_k from x_k to v, then parallel to π_{k+1} from v to x_{k+1}. It does not intersect any generating path because none of them passes beetween π_k and π_{k+1} in $P(v)$. This construction can be made for all pair of edges e_i, e_{i+1}. All the $p_{i,i+1}$ are disjointly embedded. By construction of the generating paths, the union of the $p_{i,i+1}$ do not intersect the interior of the intervals $e \cap R(e)$ $(e \in E(\mathcal{T}))$.
By claim 1, the first edge e_1 (resp. the last edge) belongs to c_1 (resp. to c_N). Therefore, the vertex $i(e_1)$ and the terminal vertex of the last edge can be connected to the extremities of the axis of D^2 by two arcs P, P' disjointly embedded in D^+. We define the curve A as $Pe_1p_{1,2}e_2 \cdots e_ip_{i,i+1}e_{i+1} \cdots P'$. By construction, it satisfies the properties of the proposition. $\diamondsuit$
This arc A will be the initial axis of the *supporting braid* (see lemma 2.16) that we construct in this section and the axis of the branched surface we will construct in section 3.

The next lemma defines the terminal axis of our *supporting braid.*

Lemma 2.12 (the accordion curve $\mathcal{R}$) *There exists an oriented curve $\mathcal{R}$ embedded in D^+ such that:*

- *The extremities of $\mathcal{R}$ are those of the axis of D^2.*
- *$\mathcal{R}$ passes through all the nailed points $y_i \in \mathcal{Y}$ and connects the rectangle $R(e_i)$ to the rectangle $R(e_{i+1})$ (where the labelling of the edges is the one given above).*
- *In each rectangle $R(e)$, $\mathcal{R}$ looks like figure 11, More precisely:*
  - *The nailed points which are the extremities of a same interval in $\varphi(\Gamma) \cap R(e)$ are consecutive along $\mathcal{R}$. The ordering of these points, according to the orientation of $\mathcal{R}$ agrees with the ordering given by the orientation of e.*
    We identify the intervals of $\varphi(\Gamma) \cap R(e)$ with the corresponding intervals along $\mathcal{R}$.
  - *The ordering of these intervals along $\mathcal{R}$ agrees with the transversal ordering given by the order $\prec$ (see section 1.1).*

This curve will be called the accordion curve.

Labelling the nailed points: The nailed points are labelled according to their ordering along the oriented curve $\mathcal{R}$.
Proof: The construction of $\mathcal{R}$ in each rectangle is given by figure 11. This construction is clearly well defined in each rectangle.
We want now to connect the last nailed point $y_j \in R(e_k)$, according to $\mathcal{R}$ in $R(e_k)$, to the first nailed point $y_{j+1} \in R(e_{k+1})$, according to $\mathcal{R}$ in $R(e_{k+1})$. Notice that y_j belongs to an incoming $P_{t(e_k)}$-side and y_{j+1} belongs to the outgoing $P_{i(e_{k+1})}$-side. By definition of the G-points, there exists a G-point in $F_i(t(e_k))$ and in $F_o(i(e_{k+1}))$. The construction of proposition 2.11 allows to connect the nailed points y_j and y_{j+1}. This construction can be used to connect any pair of such nailed points. By the argument of proposition 2.11, we obtain a curve embedded in $N(\Gamma)$ joining y_1 to $y_{\mathcal{K}}$ by connecting all these paths.
We want now to connect the nailed points y_1 and $y_{\mathcal{K}}$ to ∂D^2.

Claim 2: The first and last nailed points y_1 and $y_{\mathcal{K}}$ are exterior and accessible.

This claim is essentially a consequence of claim 1. By definition of the labelling, y_1 and $y_{\mathcal{K}}$ are exterior and can be connected to an accessible point without intersecting $\varphi(\Gamma)$ nor the axis of D^2. $\diamondsuit$
Using claim 2, we connect the nailed points y_1 and $y_{\mathcal{K}}$ to ∂D^+ by two arcs embedded in D^+, without intersecting the part of the curve $\mathcal{R}$ already constructed. $\diamondsuit$

2.2.3 Normal dissection for the supporting braid

By definition, the polygon $P(v_s)$ has an accessible corner. Thus, there exists an arc P embedded in $\overline{D^+ \setminus N(\Gamma)}$ from the terminal side to ∂D^+. We choose $\mathcal{K}$ parallel copies of this arc P connecting the initial points of the generating paths to $\mathcal{K}$ points in ∂D^+ (distincts from the extremities of the curve $\mathcal{R}$). These arcs are called the *connecting arcs.* Let $R_{\mathcal{K}}$ be the set of paths which are the concatenation of the connecting arcs with the generating paths.

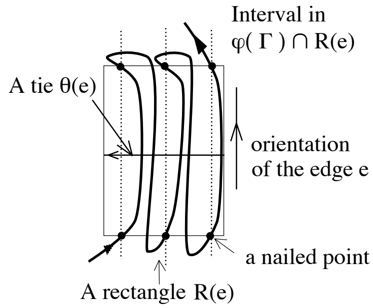


Figure 11: Accordion curve

Proposition 2.13 *The image under the homeomorphism F of the set of paths $R_{\mathcal{K}}$ is a set of disjointly embedded paths in $N(\Gamma) \cup D^+$ whose initial (resp. terminal) points are the initial points (in ∂D^+) of the paths in $R_{\mathcal{K}}$ (resp. belong to the set of nailed points $\mathcal{Y}$). They are the concatenation of two types of paths:*

- *the image of the generating paths. This is a set of disjointly embedded paths in $N(\Gamma)$ whose terminal points are the nailed points. All the intersections of $F(R_{\mathcal{K}})$ with $\varphi(\Gamma)$ are contained in the polygons $P(v)$, $v \in V(\Gamma)$.*
- *the image of the connecting arcs, called the* escape paths.

  The escape paths are oriented paths disjointly embedded in $N(\Gamma) \cup D^+$, which do not intersect $\varphi(\Gamma)$ nor the image of the generating paths. Their initial (resp. terminal) points are the initial points of the paths in $R_{\mathcal{K}}$ (resp. are the initial points of the images of the generating paths $F(\pi_i)$ $(i = 1, \cdots, \mathcal{K})$).

The proof is obvious. By construction, the paths in $R_{\mathcal{K}}$ are disjointly embedded. Since F is a homeomorphism of D_N, their images under F are disjointly embedded. Furthermore, the properties listed above are just consequences of the definition of the set $R_{\mathcal{K}}$. $\diamondsuit$

Proposition 2.14 *The escape paths $\{\tau_1, \cdots, \tau_{\mathcal{K}}\}$ are so that:*

i) They are parallel.

ii) Their intersection points with $\mathcal{R}$ either occur after the last intersection point $F(\pi_i) \cap \mathcal{R}$, or before the first intersection point $F(\pi_i) \cap \mathcal{R}$ $(i = 1, \cdots, \mathcal{K})$ according to the orientation of $\mathcal{R}$.

Proof: The escape paths are the images under F of the connecting arcs. By definition, there are $\mathcal{K}$ copies of an arc P from ∂D^2 to the terminal side $F_o(v_s)$ in $\partial N(\Gamma)$ and are embedded in $\overline{D^+ \setminus N(\Gamma)}$. The image $F(P)$ is an arc from ∂D^2 to $F(F_o(v_s))$. The escape paths are therefore $\mathcal{K}$ parallel copies of $F(P)$, proving item (i). By definition of v_s (see 2.3), $F(F_o(v_s))$ is exterior and chosen on $\partial N(\Gamma)$. As a consequence, the escape paths which are $\mathcal{K}$ copies of $F(P)$ can only intersect the accordion curve $\mathcal{R}$ outside $N(\Gamma)$, which proves item (ii). $\diamondsuit$

Remark 2.15 As we observed in remark 2.1, the homeomorphism F is defined up to a power of a Dehn twist along the core of the annulus $A = \overline{D_N \setminus N(\Gamma)}$. This information is not a priori contained in our combinatorial description of F. The choice of this Dehn twist is equivalent to a choice of a set of escape paths. Indeed, we could choose the escape paths in A going from $\partial N(\Gamma)$ to ∂D^+ and twisting around A any finite number of times.

We now consider the disc $D_{\mathcal{K}}$, whose marked points are the $\mathcal{K}$ nailed points defined above. We have all the ingredients for defining a normal dissection. Let us recall that, for a normal dissection, we identify the disc $D_{\mathcal{K}}$ with the rectangle $A_{\mathcal{K}}$ (see section 1.3.2).

Lemma 2.16 *The set of paths $\{F(R_{\mathcal{K}}); \mathcal{R}\} = \mathcal{C}$ is a normal dissection in $A_{\mathcal{K}}$, where the axis is the accordion curve $\mathcal{R}$ (see lemma 2.12). The braid $\gamma(\mathcal{C})$ in $B_{\mathcal{K}}$ given by theorem 1.11 is called the* supporting braid.

The proof is clear from the definition of a normal dissection. $\diamondsuit$

Remark 2.17

- The supporting braid is non unique. Several choices have been made during the construction: the vertex v_s, the axis $\mathcal{R}$, the generating paths and finally the escape paths. From lemma 1.14, changing any of these choices is equivalent to change the supporting braid by a conjugacy in $\mathcal{B}_{\mathcal{K}}$ or to compose it with an element of the center $\mathcal{Z}_{\mathcal{K}}$ (see remark 2.15).
- The initial and terminal axis of the supporting braid are distinct. The initial axis is the curve A given by proposition 2.11 whereas the terminal axis is the accordion curve $\mathcal{R}$. One of the main points of section 3 will be to pass from one to another.

2.3 The 2-strand block partition for the supporting braid

The aim of this subsection is to prove the following lemma:

Lemma 2.18 *The supporting braid $\gamma(\mathcal{C})$ given by lemma 2.16 admits a 2-strand block partition. Moreover, one reconstructs effectively a representative with this 2-strand block partition.*

The proof will be explicit from the construction of the normal dissection of the previous section.
We shall denote by $\mathcal{T}_s$ the tree obtained from Γ_s by removing the boundary edges $B(\Gamma_s)$ and their pre-images under ψ_s.
Let us now gather here some facts coming from our previous construction.

1. On the initial axis A:
   - The intervals beetween two points indexed by $2k-1$, $2k$ are in bijection with the edges of $\mathcal{T}_s$.
   - All the intervals coming from the same edge of $\mathcal{T}$ are consecutive along A and are ordered according to the orientation of this edge.
2. On the terminal axis $\mathcal{R}$:
   - The intervals beetween two points indexed by $2j-1$, $2j$ are in bijection with the intervals of $\varphi(\Gamma) \cap R(e)$, $e \in E(\mathcal{T})$.
   - All the intervals coming from the same rectangle $R(e)$ are consecutive along $\mathcal{R}$ and are ordered according to the ordering $\prec_e$.

2.3.1 Coding the normal dissection

Our goal, in this subsection, is to make explicit the combinatorial dissection (see 1.3.2) associated to the normal dissection $\mathcal{C} = \{F(R_{\mathcal{K}}); \mathcal{R}\}$ obtained in lemma 2.16. In order to do so, we need to find combinatorially and order (along the arcs and along the axis) the intersection points $F(R_{\mathcal{K}}) \cap \mathcal{R}$. The accordion curve $\mathcal{R}$ (lemma 2.12) is described combinatorially by listing and ordering all its intersection points with the boundary of the polygons in $N(\Gamma)$. We also have the obvious:

Remark 2.19 • The paths in $F(R_{\mathcal{K}})$ and the accordion curve can be isotoped so that all the points in $F(R_{\mathcal{K}}) \cap \mathcal{R}$ belong to $\bigcup_{v \in V(\Gamma)} P(v)$.

- By proposition 2.14, we know the intersection points of the escape paths with the accordion curve. Therefore, we only need to compute the intersection of the images of the generating paths with $\mathcal{R}$.

The combinatorial dissection will be computed from the following propositions:

Proposition 2.20 *(Intersections in a polygon $P(v)$)*
Let $P(v)$ be a polygon of $N(\Gamma)$.
The intersection points $\varphi(\pi_i) \cap \mathcal{R}$ together with their ordering along $\mathcal{R}$ and along $\varphi(\pi_i)$ are determined by the following datas:

- *the transversal word $R^{\perp}_{\varphi(\pi_i)}(e)$ and the longitudinal words $\varphi(\pi_i)$,*
- *the embedding of $\mathcal{R}$.*

See figure 13.

Proposition 2.21 *The transversal words $R^{\perp}_{\varphi(\pi_i)}(e)$ are computed from:*

- *the longitudinal words $\psi_s(e)$ $(e \in E(\mathcal{T}_s))$,*
- *the transversal words $R^{\perp}_{\pi}(e)$ $(e \in E(\mathcal{T}_s))$,*
- *the transversal words $R^{\perp}_{\pi_{\varphi_s}}(e)$ $(e \in E(\mathcal{T}))$.*

Let us recall that, if $P = \{p_1, \cdots, p_l\}$ is a set of disjointly embedded paths in $N(\Gamma)$ then $R^{\perp}_{P}(e)$ $(e \in E(\Gamma))$ denotes the transversal word at e of the paths in P (see section 1.1). Recall also that $R^{\perp}_{\pi_{\varphi_s}}(e)$ denotes the transversal words for the subdivided efficient representative (ψ_s, Γ_s).
For proving the proposition 2.20, observe first that, from the definition of the orientations, if $R^{\perp}_{\varphi(\pi_i)}(e) = I_1^{\epsilon_1} \cdots I_k^{\epsilon_k}$ $(e \in E(\mathcal{T}))$ then $I_j \cap \partial P(i(e))$ (resp. $I_{j+1} \cap \partial P(t(e))$) follows, along the outgoing $P_{i(e)}$-side (resp. along an incoming $P_{t(e)}$-side), the point $I_{j+1} \cap \partial P(i(e))$ (resp. the point $I_j \cap \partial P(t(e))$). Therefore, we obtain the cyclic ordering of all the intersection points of the boundary of any polygon with $\mathcal{R}$ and with the paths $\varphi(R_{\mathcal{K}})$. By lemma 2.6, we find the intersection points beetween the set of paths $\varphi(R_{\mathcal{K}})$ and the accordion curve $\mathcal{R}$. Then, by lemma 2.7, we obtain the respective position of these intersection points along each arc of the normal dissection and along the axis $\mathcal{R}$. $\diamondsuit$
Proof of proposition 2.21: The embedding φ induces an orientation-preserving homeomorphism then it reverses the transversal orientation in the interior of the rectangles if and only if it reverses also the longitudinal orientation. The two possible cases are illustrated in figure 12. For each path $\varphi_s(d)$ $(d \in E(\Gamma_s))$ occuring in the transversal word

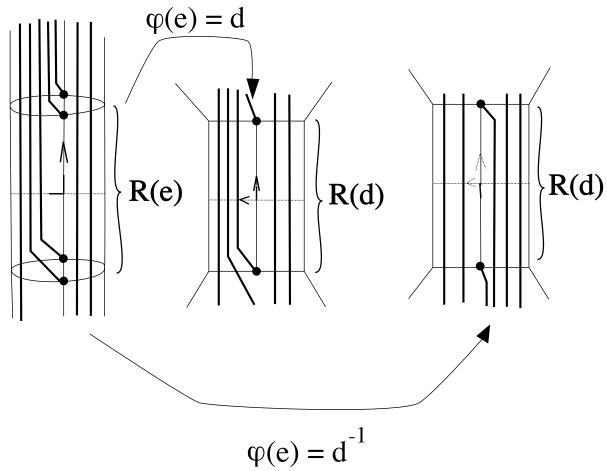


Figure 12: Computing the transversal words

$R^{\perp}_{\pi_{\varphi_s}}(e)$, there is a set of paths in the transversal word for the set $\{\varphi(\pi_i)\}_{i=1,\cdots,\mathcal{K}}$. This set is the set of the images of the generating paths passing through the rectangle $R(d)$. Their transversal position relative to another path $\varphi_s(g)$ ($g \in E(\Gamma_s)$) depends upon the transversal position of the paths $\varphi_s(d)$ and $\varphi_s(g)$ in $R(e)$. The transversal position of the paths $\varphi(\pi_i)$ arising from a same path $\varphi_s(d)$ is given by the transversal word $R^{\perp}_{\pi}(d)$. This order is preserved or reversed depending on whether φ preserves or not the transversal orientation of the edge d. $\diamondsuit$

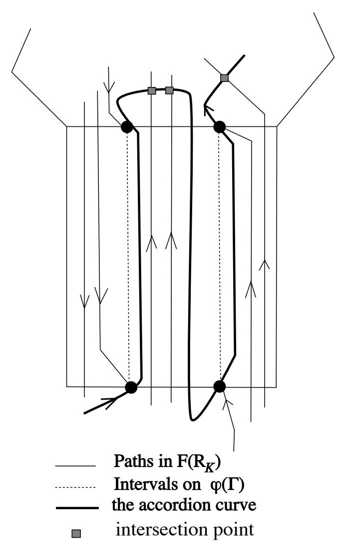


Figure 13: Intersections in the polygons

2.3.2 Block Partition

Lemma 2.22 *Two consecutive paths P_{2k-1}, P_{2k} in $F(R_{\mathcal{K}})$ intersects $\mathcal{R}$ the same number of times and the i^{th} intersection points of P_{2k-1} and P_{2k} with the axis are consecutive along $\mathcal{R}$.*

Proof: Notice that a G-point which belongs to a polygon of a boundary vertex has always an odd index.
Recall that, in a normal dissection, the paths are labelled according to the labelling of their initial points. From proposition 2.14, it is sufficient to show the lemma for the images of the corresponding generating paths, i.e. those of indices $2k-1$ and $2k$. The generating paths π_{2k} and π_{2k-1} go through the same collection of rectangles and polygons in $N(\Gamma_s)$, except for the last rectangle intersected by π_{2k-1}. By construction, π_{2k-1} and π_{2k} are parallel and so are their images under φ. Therefore, the intersections of P_{2k-1} and P_{2k} with $\mathcal{R}$ satisfy the properties of the lemma, except perhaps in the image of the last rectangle of $\mathcal{T}_s$ intersected by π_{2k-1} and π_{2k}. In the last rectangle also, the paths P_{2k-1} and P_{2k} intersect $\mathcal{R}$ in the same way, by definition of the accordion curve, as shown by figure 14 (the figure shows the only two possible models for the intersection with the last rectangle). This completes the proof. $\diamondsuit$

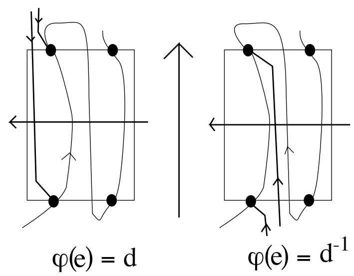


Figure 14: Lemma 2.22

We can now give the
Proof of lemma 2.18: The proof follows from lemmas 2.22 and 1.13. In lemma 1.13, the normal dissection is reduced, which is not necessarily the case for the normal dissection $\mathcal{C}$ of lemma 2.16. In order to transform the normal dissection $\mathcal{C}$ to a reduced one, we have to suppress the trivial discs. There are two types of trivial discs (see section 1.3.2 and figure 4). When a path P_{2k-1} bounds a trivial disc of type I, then by lemma 2.22, the path P_{2k} also bounds such a trivial disc (and conversally). So, after suppressing all the trivial discs of type I, the conclusion of lemma 2.22 is again satisfied. The supression of a trivial disc of type II removes exactly one intersection point of a given arc with the axis $\mathcal{R}$. So the unique reduced normal dissection in the class of $\mathcal{C}$ satisfies the assumptions of lemma 1.13. This completes the proof. $\diamondsuit$
Let us now summarize our algorithm for the construction of the supporting braid:

Data: A pseudo-Anosov braid β
Let $[f_\beta] \in MCG(D_N)$ be the induced isotopy class.
Step 1: Apply the train-track algorithm ([BH1],[Lo1],[FM]) to obtain a canonical efficient representative (ψ, Γ).

`The supporting braid algorithm`

Data: The longitudinal and transversal words of the efficient representative (ψ, Γ).
Step 2:
Choice of a vertex v_s (see proposition 2.3). Orientation of the tree $\mathcal{T} = \Gamma \setminus B(\Gamma)$ toward v_s.
Choice of a set of generating paths.

Step 3: Compute the longitudinal and transversal words for the set of images of the generating paths.
These paths are obtained from the efficient representative.
Step 4: Compute the accordion curve $\mathcal{R}$ (see lemma 2.12).
Step 5: Compute the normal dissection $\mathcal{C}$ by using propositions 2.20, 2.21, 2.14. Reduce $\mathcal{C}$.
Step 7: Reconstruction of the supporting braid (section 1.3.2 and proposition 1.13).

3 Periodic orbits of the suspension flow

3.1 Preliminaries: Coding, Branched Surfaces

Let β be a pseudo-Anosov braid in $\mathcal{B}_N$ and let $[f_\beta]$ be the associated isotopy class in $MCG(D_N)$.
Let (ψ, Γ) be a canonical efficient representative of an isotopy class $[\phi]$ in the conjugacy class of $[f_\beta]$ and let (ψ_s, Γ_s) be the corresponding subdivided efficient representative.
Let γ be the supporting braid constructed in the previous section. We recall that γ is a braid in $\mathcal{B}_{\mathcal{K}}$, where $\mathcal{K}$ depends only upon (ψ, Γ) and has been given in proposition 2.5.
Let us recall that F denotes a pseudo-Anosov homeomorphism of D_N, such that $[F] = [\phi]$ (see section 2.1). Moreover, recall that F is only defined up to a power of H_Δ (see remark 2.1).
We recall that the suspension flow $(F_t^\beta)_{t \in \mathbf{R}}$ associated to β is a suspension flow of an element of $[f_\beta]$ which has $\overline{\beta}$ (and not $\overline{\beta \Delta_N^k}$) as suspension of the periodic orbits of the punctures.
Our main goal now is to give an effective way to embed in M_β^3 any periodic orbit of this flow given by a *symbolic coding*. We recall briefly the basic notions we need for coding the periodic orbits of the pseudo-Anosov homeomorphism F.
A pseudo-Anosov homeomorphism admits a Markov partition (see [FLP], chap.10) and from a Markov partition, one gets a symbolic coding (see [Sh] for instance). One of the main advantage of the efficient representative is the fact that we can easily find (by an effective process) such a Markov partition (see [BH1] and [Lo2] for details). In particular, it is proved in [Lo2] that, except perhaps a finite set, all the periodic orbits of the pseudo-Anosov homeomorphism F are given by a coding whose rectangles are in 1-1 correspondance with the edges of the subdivided graph Γ_s. The possible exceptions are the *singular orbits*, i.e. the orbits of the singularities of the invariant foliation. Furthermore, even the singular orbits can be found from the efficient representative. The other orbits will be called *regular*.
In what follows, according to [Lo2], we will not distinguish beetween the regular periodic orbits of the homeomorphism F and the periodic orbits of the map ψ_s whose points belong to the interior of the edges of Γ_s.
Let us recall that $\mathcal{T}_s$ is the tree obtained from Γ_s by removing the boundary edges $B(\Gamma_s)$ and collapsing their pre-images under ψ_s. We will only consider regular periodic orbits of ψ_s whose points belong to the interior of the edges of $\mathcal{T}_s$. This restriction does not suppress any periodic orbit. Indeed, the orbits of the punctures still exist as the orbits of the end points of $\mathcal{T}_s$.

Definition 3.1
- The *symbolic coding* $\sum_{(\psi_s, \mathcal{T}_s)}$ is the set of (infinite) sequences $(\cdots e_{i_j} e_{i_{j+1}} \cdots)$ ($j \in \mathbf{Z}$) such that the symbols e_{i_j} are in bijection with the edges of $\mathcal{T}_s$.
- A sequence $\cdots e_{i_1} \cdots e_{i_k} e_{i_{k+1}} \cdots$ is *admissible for* $\sum_{(\psi_s, \mathcal{T}_s)}$ if, for all $j \in \mathbf{Z}$, one has $e_{i_{j+1}}^{\pm 1} \subset \psi_s(e_{i_j})$.

- An admissible sequence is *periodic* if it has the form $(e_{i_1} \cdots e_{i_k})^\infty$ where the symbol $(.)^\infty$ means that the (finite) sequence $(.)$ is repeated indefinitely.

- The smallest such k is called the *length* of the admissible periodic sequence.

An admissible periodic sequence of length k defines k finite admissible periodic sequences of length k, which differ only by a cyclic permutation of the letters (a shift). All these sequences defines the same periodic orbit of F.
It will sometimes be convenient for our purpose to distinguish these sequences. In that case, we will omit the symbol $(.)^\infty$ and keep only the finite sequence.
We are going first to construct a *branched surface* in $D^2 \times [0,1]$ from the supporting braid defined in the previous section. Our first aim will be to prove that, when identifying the disc $D^2 \times \{0\}$ with the disc $D^2 \times \{1\}$, this branched surface is the one we are looking for.
Let us start by a brief review on branched surfaces. We refer the interested reader to [W73], [BW] or [GHS].
A *branched surface* W is a finite two dimensional CW-complex embedded in a compact oriented 3-manifold M such that W is a manifold except at a subset of its 1-skeleton, called the *branch locus of* W. A 2-dimensional tangent space is defined at each point of W.

Definition 3.2 • A *branched surface with bands* W is a branched surface satisfying:

i) The branch locus B is a union of disjoint intervals I_i $(i = 1, \cdots, k)$. The neighborhood of the I_i is shown in figure 15. In this figure, we have distinguished the two sides of the tangent plane in such a neighborhood, namely the locally one-sheeted side and the locally k-sheeted side.

ii) The complement of B in W is a collection of rectangles $[0,1]\times[0,1]$, glued together along their parallel sides $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$, to the neighborhoods of the I_i as shown in figure 15. The rectangles are the *bands* of W. The *top* (resp. *bottom*) of a band is its side $[0,1] \times \{0\}$ (resp. $[0,1] \times \{1\}$) attached on the locally 1-sheeted side (resp. locally k-sheeted side) of an interval in the branch locus.

- A *braided branched surface* is a branched surface with bands embedded in the complement of a closed braid in the solid torus. All the intervals of its branch locus lie on an axis in a single fiber (say $D^2 \times \{0\}$), called the *axis of the branched surface.* The bands are transverse to the fibers $D^2 \times \{\theta\}$, $\theta \in \mathbf{S}^1$.

The branched surfaces with bands that we defined above are very similar to the templates of [GHS]. The only difference is that the locally k-sheeted side of an interval in the branch-locus is always a locally 2-sheeted side ($k = 2$) for a template.
In this section, we are interested in a subclass of the *braided branched surfaces*, called *special* for a closed braid $\overline{\beta}$, The braid $\beta \in \mathcal{B}_N$ is supposed to be pseudo-Anosov.

Definition 3.3 A branched surface $W_{\overline{\beta}}$ is special for $\overline{\beta}$ if:

iii) $W_{\overline{\beta}}$ is a braided branched surface embedded in M^3_{β}, some of its boundary components being the closed braid $\overline{\beta}$.

iv) Each closed braid *carried* by $W_{\overline{\beta}}$ is ambient isotopic to a periodic orbit of the flow ϕ^α_t defined as follows:

- ϕ_t^α is a suspension flow of a pseudo-Anosov element in $[f_\alpha] \in MCG(D_N)$, where α is conjugated to β in $\mathcal{B}_N$.
- The flow ϕ_t^α is given so that the orbits of the marked points under the flow in $D^2 \times S^1$ are exactly the braid $\overline{\alpha}$ (and not $\overline{\alpha\Delta^k}$, $\Delta \in \mathcal{Z}_N$).

A braided branched surface W in M^3_β *carries* a closed braid $\overline{\alpha}$ if the link $\overline{\alpha}$ is embedded in W such that its intersection with the bands of W is a collection of paths disjointly embedded in the interior of these bands and oriented from the top to the bottom (see definition 3.2 ii)).

Remark 3.4 The definition of a branched surface with bands implies that it carries a semi-flow. In particular, each band has exactly one top and one bottom.

Let W be a braided branched surface in M^3_β. We define now a branched surface with bands W' embedded in $D^2 \times [0,1] \setminus \beta$. To this end, we cut open M^3_β along the fiber D_0 which contains the branch locus B. The result is a collection of open rectangles embedded in $D^2 \times (0,1) \setminus \beta$. Then we compactify $D^2 \times (0,1)$ by adding two copies of D_0, one at $D^2 \times \{0\}$, the other at $D^2 \times \{1\}$. We then recover the branching at $D^2 \times \{1\}$ (see figures 15 and 21). This branched surface W' *carries* a representative of a braid α if and only if $\overline{\alpha}$ is carried by W. A braid θ with a 2-strand block partition is a particular such branched surface W', where the branch locus is empty. Indeed, any pair of strands $(2i-1, 2i)$ of θ, together with the intervals on the initial and terminal axis connecting the initial and terminal points of these strands bound an embedded rectangle in the solid cylinder. From the properties of the 2-strand block partition, all these rectangles are disjointly embedded. Therefore, it makes sense for θ to *carry* a braid.

Definition 3.5 A symbolic coding $\sum_{(\psi_W, B_W)}$ associated to a braided branched surface W is defined in the following way:

- The symbols in $B_W = \{I_1, \cdots, I_k\}$ are in bijection with the tops of the bands of the branched surface.
- The map ψ_W is a piecewise continuous function defined on the intervals I_i. It is strictly monotone on each interval. The image of an interval I_i under ψ_W is the union of the intervals in B_W, which are the bottom of the band whose I_i is the top.

A closed braid with one component which is carried by W defines a unique, up to shift, periodic admissible sequence for $\sum_{(\psi_W, B_W)}$. The following proposition shows that the converse is also true.

Proposition 3.6 *Let W be a braided branched surface. Any periodic admissible sequence for $\sum_{(\psi_W, B_W)}$ defines a unique closed braid with one component carried by W.*

<u>**Proof:**</u> A periodic admissible sequence for $\sum_{(\psi_W, B_W)}$ defines a finite collection of points in the interior of the intervals of the branch locus. These points are periodic points for the map ψ_W. They are ordered along the branch locus by applying the kneading theory as defined by [CE] (see section 3). As a consequence, if $I \in B_W$ contains k points of this orbit, then $\psi_W(I)$ contains also k points. There is only one way to connect these points by k disjointly embedded paths in this band. Therefore, we obtain a unique closed braid carried by W. $\diamondsuit$

A theorem of [Lo2] asserts that a special braided branched surface exists for all pseudo-Anosov braids. We give here a constructive proof of this theorem. In addition, our construction gives explicitly the embedding of the branched surface.
As a consequence, we obtain an effective way for embedding the periodic orbits of the suspension-flow of the homeomorphism F, using this braided branched surface.

3.2 The special braided branched surface for $\overline{\beta}$

The supporting braid $\gamma(\mathcal{C})$ has a 2-strand block partition by lemma 2.18. Its strands are denoted b_i $(i = 1, \cdots, \mathcal{K})$ and the labelling is the one of section 2. The braid $\gamma(\mathcal{C})$ is embedded in $D^2 \times [0,1]$. Its initial (resp. terminal) points in $D^2 \times \{0\}$ (resp. $D^2 \times \{1\}$) are in bijection with the G-points (resp. nailed points) given by definition 2.4. Its initial axis in $D^2 \times \{0\}$ is the curve A given by proposition 2.11 whereas its terminal axis in $D^2 \times \{1\}$ is the accordion curve $\mathcal{R}$ of lemma 2.12.
The braid $\gamma(\mathcal{C})$ is obtained from the normal dissection $\mathcal{C}$ in $D_{\mathcal{K}}$ whose axis is the accordion curve $\mathcal{R}$. Let T_t $(t \in [0,1])$ be an isotopy of D^2, which fixes ∂D^2 pointwise, and so that T_0 is the identity and $T_1(\mathcal{R})$ is the x-axis of D^2.
Recall that we have denoted I_i $(i = 1, \cdots, \mathcal{K}/2)$ the intervals on $\mathcal{R}$ which connect the nailed points of indices $2k-1$ and $2k$. From our labelling and the definition of the accordion curve, the intervals $T_1(I_i)$ and $T_1(I_{i+1})$ are consecutive along the x-axis. Let us denote by $I(e)$ the subset of $\{T_1(I_i)\ ,\ i = 1, \cdots, \mathcal{K}/2\}$ on the x-axis so that $T_1(I_j) \in I(e)$ if $I_j \in \mathcal{R} \cap R(e)$, where $e \in E(\mathcal{T})$. Two intervals $T_1(I_j)$ and $T_1(I_{j+1})$ in $I(e)$ satisfy $I_j \prec_e I_{j+1}$.
Since $\gamma(\mathcal{C})$ has a 2-strand block partition, then the two strands b_{2j-1}, b_{2j} of initial points x_{2j-1}, x_{2j} have their terminal points at two marked points y_{2k-1}, y_{2k} in $D^2 \times \{1\}$. They are the boundary of an interval $T_1(I_i)$ in some $I(e)$.
The intervals J_j which connect the points x_{2j-1} and x_{2j} on $D^2 \times \{0\}$ are in 1-1 correspondence with the edges of the subdivided tree $\mathcal{T}_s$. Furthermore, their labelling and their respective position along the x-axis agree with the orientation of the corresponding edge e of $\mathcal{T}$. Let us denote by $J(e)$ the set of intervals J_i on the x-axis of $D^2 \times \{0\}$ which corresponds to the same edge $e \in E(\mathcal{T})$.

Definition 3.7 Let (ψ, Γ) be a canonical efficient representative of $[f_\beta] \in MCG(D_N)$ and let $\gamma(\mathcal{C})$ be the supporting braid of lemma 2.18.
We denote by $\overline{W_{(\psi,\Gamma)}}$ the branched surface constructed as follows:

i) A band B_{2j-1} of $\overline{W}_{(\psi,\Gamma)}$ is a rectangle embedded in $D^2 \times [0,1]$ whose boundary if formed by:

- the two strands b_{2j-1}, b_{2j} of $\gamma(\mathcal{C})$.
- the interval J_j on the x-axis of $D^2 \times \{0\}$ and the interval $T_1(I_i)$ on the x-axis of $D^2 \times \{1\}$, whose boundary points are the terminal points of b_{2j-1}, b_{2j}.

ii) The branch locus is defined as follows:

a) We identify in $D^2 \times \{1\}$ all the intervals of $I(e)$ to a single interval $\mathcal{I}(e)$ on the x-axis of $D^2 \times \{1\}$. The identification is made in $D^2 \times [0,1]$ by declaring that $T_1(I_i) \in I(e)$ is *identified over* $T_1(I_{i-1}) \in I(e)$. This means that the two strands which end at $\partial(T_1(I_i))$ cross over the two strands ending at $\partial(T_1(I_{i-1}))$. The same property holds for the corresponding rectangles (see (i)).

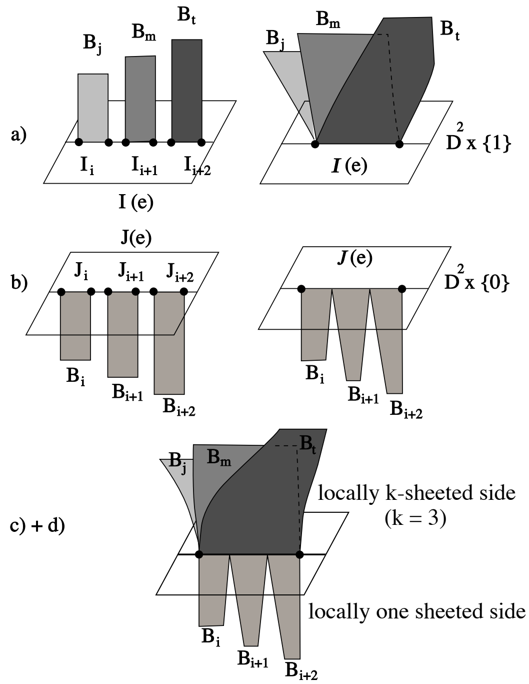


Figure 15: Construction of $\overline{W}_{(\psi,\Gamma)}$

b) The intervals of $J(e)$, i.e. $J_j, J_{j+1}, \cdots, J_{j+k}$, are glued together in $D^2 \times \{0\}$ by identifying the last point of J_{j+n} with the first point of J_{j+n+1} $(0 \leq n \leq k-1)$. The resulting interval on the x-axis is denoted $\mathcal{J}(e)$. The branched surface obtained at this point by *smoothing the branching* (see step d)) is denoted $W_{(\psi,\Gamma)}$.

c) We identify the two discs $D^2 \times \{0\}$ and $D^2 \times \{1\}$ without twisting so that the intervals $\mathcal{I}(e)$ and $\mathcal{J}(e)$ corresponding to the same edge e of $\mathcal{T}$ are identified.

d) At a given interval $\mathcal{I}(e) \simeq \mathcal{J}(e)$, we smooth the branching in a way that the tangent plane of all the rectangles glued together agree.

This definition is illustrated by figure 15.
We want to prove the following

Theorem 3.8 *Let β be a pseudo-Anosov braid in $\mathcal{B}_N$ and let $[f_\beta] \in MCG(D_N)$ be the corresponding isotopy class.*
Let (ψ, Γ) be a canonical efficient representative of $[f_\beta]$.
Then the branched surface $\overline{W}_{(\psi,\Gamma)}$ of definition 3.7 is a special braided branched surface for the closed braid $\overline{\beta}$.

Let us start the proof by:

Proposition 3.9 *Let β be a pseudo-Anosov braid and (ψ, Γ) be a canonical efficient representative for $[f_\beta]$.*
Then some boundary components of the branched surface $\overline{W}_{(\psi,\Gamma)}$ are ambient isotopic in M^3_β to the closed braid $\overline{\beta}$.

Proof: Let $g_1, \cdots, g_N$ be the N G-points which belong to the $P_{v_{b_j}}$-sides, where the v_{b_j} are the boundary vertices (see section 1.2). These boundary vertices are periodic under F and so are the $g_1, \cdots, g_N$. These points lie in the boundary of $\overline{W}_{(\psi,\Gamma)}$ in the fiber $D^2 \times \{0\}$. Therefore, the N paths in $R_{\mathcal{K}}$ ending at $g_1, \cdots, g_N$ give rise, by our construction, to a closed braid in the boundary of $\overline{W}_{(\psi,\Gamma)}$. This closed braid is ambient isotopic in M^3_β to a suspension under $(F^\beta_t)_{t\in\mathbf{R}}$ of $g_1, \cdots, g_N$ if the homeomorphism is well chosen by lemma 1.14. These points have the same trajectories under $(F^\beta_t)_{t\in\mathbf{R}}$ than the punctures. Therefore, the closed braid is ambient isotopic in M^3_β to $\overline{\beta}$. $\diamondsuit$

Remark 3.10 In order to fix the homeomorphism, we have to choose a power of H_Δ. By remark 2.15, this choice is made by choosing the escape paths.
For this last choice, we only have to compare $\overline{\beta}$ with $\overline{\beta\Delta^k_N}$ for some $k \in \mathbf{Z}$. This can be done, for instance, by comparing the algebraic lengths of the braids β and $\beta\Delta^k_N$ in the generators of $\mathcal{B}_N$.

Proposition 3.11 1. *The branched surface $\overline{W}_{(\psi,\Gamma)}$ is a braided branched surface in M^3_β.*

2. *If $J \in J(e)$ and $I \in \mathcal{I}(e')$ are respectively the top and the bottom of the same band of $\overline{W}_{(\psi,\Gamma)}$, then the edges $e \in E(\mathcal{T}_s)$ and $e' \in E(\mathcal{T})$ satisfy $e'^{\pm 1} = \psi_s(e)$.*

__Proof:__ By construction, $\overline{W}_{(\psi,\Gamma)}$ is a braided branched surface. Indeed, the branch locus is a disjoint collection of intervals embedded in a single fiber $D^2 \times \{0\} = D^2 \times \{1\}$. Its complement is, by construction, a collection of rectangles disjointly embedded and transverse to the fibers. Furthermore, $\overline{W}_{(\psi,\Gamma)}$ is embedded in M^3_β by proposition 3.9 (see remark 3.10). This proves part 1).
The intervals of the branch locus are in bijection with the edges of $\mathcal{T}$. The intervals $J_i \in J(e)$ in $D^2 \times \{0\}$ are in bijection with the edges of $\mathcal{T}_s$ contained in $e \in E(\mathcal{T})$. The bands of the branched surface are in bijection with the 2-strand blocks of the supporting braid. Item 2/ is straightforward from the definition of the supporting braid and the part ii-a) of definition 3.7. $\diamondsuit$

Proposition 3.12 *Let F be a pseudo-Anosov homeomorphism in the class $[f_\beta]$ and let $\sum_{(\psi_s,\mathcal{T}_s)}$ be the symbolic coding for F given by definition 3.1. If $\sum_{\overline{W}_{(\psi,\Gamma)}}$ is the symbolic coding associated to the branched surface $\overline{W}_{(\psi,\Gamma)}$ of definition 3.7, then any admissible sequence for $\sum_{\overline{W}_{(\psi,\Gamma)}}$ is admissible for $\sum_{(\psi_s,\mathcal{T}_s)}$ and the converse holds.*
In particular, any closed braid with one component carried by the branched surface $\overline{W}_{(\psi,\Gamma)}$ defines a unique periodic admissible sequence for $\sum_{(\psi_s,\mathcal{T}_s)}$ and therefore a periodic orbit of F. The converse is also true.

__Proof:__ The correspondence beetween the two symbolic codings is straightforward from proposition 3.11 2). Any closed braid with one component carried by the branched surface defines a unique periodic admissible sequence for $\sum_{\overline{W}_{(\psi,\Gamma)}}$ by definition of this coding. The correspondence beetween the two codings implies then that any closed braid carried by $\overline{W}_{(\psi,\Gamma)}$ defines a unique coding for $\sum_{(\psi_s,\mathcal{T}_s)}$. The second part of proposition 3.12 is straightforward from proposition 3.6. $\diamondsuit$

We are now going to prove the following lemma whose proof relies on lemmas and propositions 3.14 to 3.19.

Lemma 3.13 *Let $\overline{\alpha}$ be a closed braid with one component carried by $\overline{W}_{(\psi,\Gamma)}$ and let P be the corresponding periodic orbit of F (see proposition 3.12). Then $\overline{\alpha}$ is ambient isotopic in M^3_β to the orbit $\{F^\beta_t(x)\}_{t\in\mathbf{R}}$ $(x \in P)$ of the suspension of F.*

The accordion curve $\mathcal{R}$ is the terminal axis of the supporting braid whereas the curve A given by proposition 2.11 is its initial axis. They are embedded in two different discs $D^2 \times \{0\}$ and $D^2 \times \{t\}$ ($t > 0$). It will be convenient for our purpose to consider them embedded in a single disc.
If P is a periodic orbit of the homeomorphism F, then there is at most one point of P on a given tie of the fibered neighborhood $N(\Gamma)$ (see [Lo2] for instance). Moreover, each point of P belongs to the interior of a unique rectangle $R(e)$, $e \in E(\mathcal{T})$, and therefore to the interior of a unique rectangle $F(R(e')) = \varphi_s(R(e'))$, $e' \in E(\mathcal{T}_s)$. Each of these last rectangles contains exactly one interval of $\varphi_s(\Gamma) \cap R(e)$. We have identified these intervals $\varphi_s(\Gamma) \cap R(e)$ with the intervals on the accordion curve connecting their boundary points in $\partial R(e)$. We project the points of P along the ties on their corresponding intervals on the accordion curve. This gives a collection of points P' in bijection with the points of P and ordered along the accordion curve. Then we consider the axis A' which goes through the points of P' in the order of their projection on the edges of $\mathcal{T}_s$. The axis A' is clearly homotopic (relative to ∂D^2) to the axis A going through the projection of P' on $\mathcal{T}_s$. In what follows, we won't distinguish A and A'.
The points of P are labelled according to their position along the axis A.
In what follows (propositions 3.14, 3.15, 3.16), we give a construction of all the closed braids carried by $\overline{W}_{(\psi,\Gamma)}$, starting from any finite collection of admissible periodic sequences for $\sum_{(\psi_s,\mathcal{T}_s)}$.

Proposition 3.14 *Let $\sigma_1, \cdots, \sigma_r$ be any finite collection of periodic admissible sequences for $\sum_{(\psi_s,\mathcal{T}_s)}$, let $Q = \{Q_1, \cdots, Q_l\}$ be the corresponding periodic orbits for ψ_s of periods $q_1, \cdots, q_l$ and let $m = q_1 + \cdots + q_l$.*
Let $R_\mathcal{K}$ be the set of paths from ∂D^+ to the G-points defined in section 2.2.
These paths are ordered by the relation $\prec_{v_s}$ (see section 2).
Then, there exists a set of m paths $B_P = \{P_1, \cdots, P_m\}$ in D^+ connecting m points in ∂D^+ to the points x_i of the periodic orbits. They satisfy the following properties:

- *The paths in $B_P \cup R_\mathcal{K}$ are disjointly embedded.*
- *Let x_j be in e_{i_j} ($e_{i_j} \in E(\mathcal{T}_s)$) and let P_j be the path in B_P ending at x_j. If π_i and π_{i+1} are the two generating paths whose G-point belong to the edge e_{i_j}, then $\pi_i \prec_{v_s} P_j \prec_{v_s} \pi_{i+1}$.*
- *Let x_i, x_j be two periodic points in Q which belong to a same edge of $\mathcal{T}$ and such that x_j follows x_i according to the orientation of the edge. Then P_j follows P_i according to the order $\prec_{v_s}$.*
- *Let x_i, x_j be two periodic points in Q which belong to two distinct edges e_k and e_n of $E(\mathcal{T}_s)$ whose indices satisfy: $k < n$. Then $P_i \prec_{v_s} P_j$ holds.*

Proof: For proving this result, we construct the paths $\{P_1, \cdots, P_m\}$ parallel to the generating paths. The construction is exactly the same as in definition 2.8. $\diamondsuit$

Proposition 3.15 *With the above notations, the normal dissection $\{F(B_P); \mathcal{R}\}$ defines a braid $\alpha_B(\alpha)$ carried by the supporting braid $\gamma(\mathcal{C})$ of lemma 2.18.*

Proof:

1. The construction of section 2 applied to the set of paths $R_\mathcal{K} \cup B_P$ defines a normal dissection $\mathcal{C}_P$ whose arcs are the images under F of the set of paths $R_\mathcal{K} \cup B_P$ and whose axis is the accordion curve of lemma 2.12.

2. The normal dissection $\mathcal{C}_P$ is such that any arc in $F(B_P)$ of $\mathcal{C}_P$ lies beetween and is parallel to the two arcs in $F(R_\mathcal{K})$ which form a block of the supporting braid $\gamma(\mathcal{C})$ of lemma 2.18.

In the normal dissection $\mathcal{C}_P$, we consider the sub-dissections $\{F(R_\mathcal{K}); \mathcal{R}\}$ and $\{F(B_P); \mathcal{R}\}$. The first one gives rise to the supporting braid $\gamma(\mathcal{C})$. The second one gives rise to a braid $\alpha_B(\alpha)$ carried by the supporting braid. $\diamondsuit$
From the braid $\gamma(\mathcal{C}_P)$, one reconstructs in a single operation:

- The braided branched surface $\overline{W}_{(\psi,\Gamma)}$ of definition 3.7, and
- A braid α whose closure is a link with l components which is carried by $\overline{W}_{(\psi,\Gamma)}$.

The construction goes as follows:
The first step is to construct the branched surface $W_{(\psi,\Gamma)}$. In definition 3.7, this is done by identifying the bottoms of some rectangles given by the supporting braid.
After this identification, the strands of $\alpha_B(\alpha)$ give rise to a braid α carried by $W_{(\psi,\Gamma)}$. Indeed, no end-points of $\alpha_B(\alpha)$ can be identified by the above operation since the points belong to a finite collection of distinct periodic orbits of F. Therefore, the strands of α are well defined. By this construction, one obtains a representative of α carried by the branched surface $W_{(\psi,\Gamma)}$.
The second point is to construct the branched surface $\overline{W}_{(\psi,\Gamma)}$ from $W_{(\psi,\Gamma)}$ (see definition 3.7). By the above argument, the closed braid $\overline{\alpha}$ has the same number of components as the number of periodic orbits and is carried by $\overline{W}_{(\psi,\Gamma)}$.

The above construction will be called the *braid builder construction.*

Proposition 3.16 *Any closed braid carried by the branched surface $\overline{W}_{(\psi,\Gamma)}$ is obtained by the braid builder construction.*

__Proof:__ We suppose that the branched surface $\overline{W}_{(\psi,\Gamma)}$ (see definition 3.7) is given, as well as a closed braid $\overline{\alpha}$ carried by $\overline{W}_{(\psi,\Gamma)}$. This $\overline{\alpha}$ defines a unique collection of periodic orbits P of the homeomorphism F (see proposition 3.12). Now, the braid builder construction applied to this collection defines a closed braid carried by $\overline{W}_{(\psi,\Gamma)}$. It has the same symbolic coding (for $\sum_{W_{(\psi,\Gamma)}}$ or $\sum_{(\psi,\mathcal{T}_s)}$) as $\overline{\alpha}$. By uniqueness of the closed braid defined by a given periodic admissible sequence, the two braids are the same. $\diamondsuit$
The following lemma shows that all the braids carried by $W_{(\psi,\Gamma)}$ split in a well defined way.

Lemma 3.17 *Let α be any braid carried by $W_{(\psi,\Gamma)}$.*
It splits as $\alpha = \alpha_B(\alpha)\alpha_S(\alpha)$ where:

1. *The braid $\alpha_B(\alpha)$ is the braid given by proposition 3.15.*
2. *The braid $\alpha_B(\alpha)$ has a block-partition. The blocks of $\alpha_B(\alpha)$ are in bijection with the blocks of the supporting braid which carry (as a 2-strand block) at least one strand of α.*

   For the writing of $\alpha_B(\alpha)$ as a word in the block letters, the other blocks of the supporting braid are deleted.
3. *The initial axis of $\alpha_S(\alpha)$ is the terminal axis of $\alpha_B(\alpha)$, i.e. the accordion curve $\mathcal{R}$. The i^{th} strand of $\alpha_S(\alpha)$ is the one starting from the i^{th} point along this axis.*

i) All the crossing-letters of $w(\alpha_S(\alpha))$ are of the form $s_{i,j}^{-1}$, with $i > j$.

ii) The crossing-letter $s_{i,j}^{-1}$ $(i > j)$ occurs in the crossing-word $w(\alpha_S(\alpha))$ if and only if the respective positions, along $\mathcal{R}$, of the terminal points of the strands i and j do not agree with their ordering along the axis A (see figure 16).

Proof: By proposition 3.16, α is constructed by the braid builder construction. This construction has two parts. First, it gives the normal dissection $\{F(B_P); \mathcal{R}\}$. The second step goes from the supporting braid to the branched surface $W_{(\psi,\Gamma)}$ by identifying the accordion curve $\mathcal{R}$ with the axis A (see step ii-a) of definition 3.7). The first step gives rise to $\alpha_B(\alpha)$ which is carried by the supporting braid. The second step gives rise to the braid $\alpha_S(\alpha)$. This proves the splitting property. Item 1) is then obvious. Item 2) comes from the fact that $\alpha_B(\alpha)$ is carried by the supporting braid $\gamma(\mathcal{C})$ which has a 2-strand block partition. The item 3-i) comes from the way the intervals $I(e)$ have been identified at the step ii-a) of definition 3.7. More precisely, the intervals $I(e)$ along the accordion curve are identified from the right over the left. Therefore, all the crossings are of the form right crosses over left. Thus the corresponding crossing-letter is of the form $s_{i,j}^{-1}$, with $i > j$. The consequence is that the crossing letter $s_{i,j}^{-1}$, $i > j$ belongs to $w(\alpha_S(\alpha))$ if and only if the ordering of the points i and j do not agree along the two axis $\mathcal{R}$ and A. For proving item 3-ii), it suffices to look at the permutation induced by $\alpha_S(\alpha)$. One implication is clear: if the points i and j are permuted from $\mathcal{R}$ to A, then the letter $s_{i,j}^{-1}$ must occur in $w(\alpha_S(\alpha))$. On the contrary, suppose $s_{i,j}^{-1}$, $i > j$ occurs in $w(\alpha_S(\alpha))$. Then the respective position of the points i and j is reversed from $\mathcal{R}$ to A. Indeed, the crossings $s_{i,j}$, $i > j$ or $s_{j,i}^{-1}$ do not occur in $\alpha_B(\alpha)$. Therefore, the permutation induced by $s_{i,j}^{-1}$ remains. This proves 3-ii). $\diamondsuit$

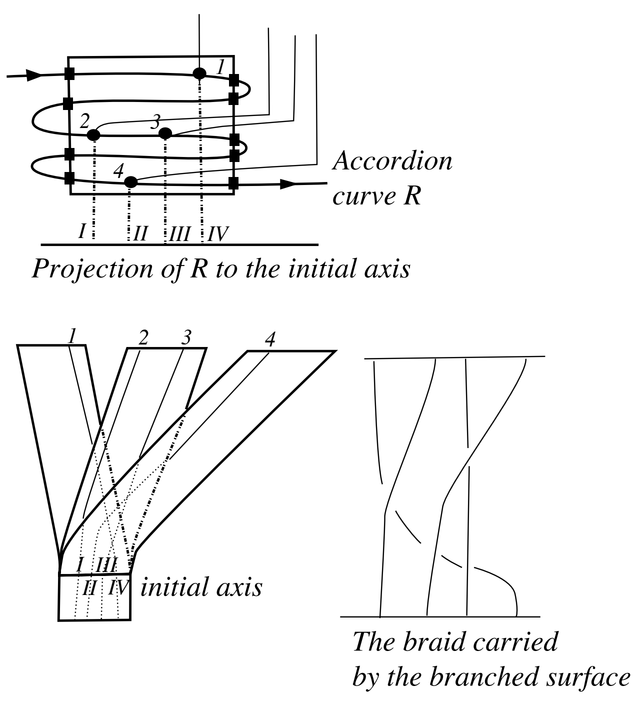


Figure 16: Lemma 3.17-3

Proposition 3.18 *Let $\overline{\alpha}$ be a closed braid with one component carried by $\overline{W}_{(\psi,\Gamma)}$. Let P be the periodic orbit of F defined from $\overline{\alpha}$ by proposition 3.12. The braid $\alpha_B(\alpha)$ of lemma*

3.17 is equivalent to a braid $\theta(\alpha)\gamma_0$ where $\overline{\gamma_0}$ is ambient isotopic in M_β^3 to a suspension of P under $(F_t^\beta)_{t\in\mathbf{R}}$.

Proof: The braid $\alpha_B(\alpha)$ has been constructed from the normal dissection $\{F(B_P);\mathcal{R}\}$ where

- the terminal points of the paths in B_P are the points of P.
- $\{B_P;A\}$ is a trivial normal dissection, where A is the initial axis of the supporting braid defined in proposition 2.11.

This implies, by lemma 1.14, that the closure of $\alpha_B(\alpha)$ is ambient isotopic to the suspension of P under $(F_t^\beta)_{t\in\mathbf{R}}$ if $\{B_P;\mathcal{R}\}$ is a trivial dissection (in that case, $\theta(\alpha)$ is the trivial braid). Indeed, recall that the choice of the suspension flow was done in proposition 3.9 (see remark 3.10).
If $\{B_P;\mathcal{R}\}$ is non trivial, we denote by $\theta(\alpha)$ the braid corresponding to the dissection $\{B_P;\mathcal{R}\}$. Let f_θ be an homeomorphism of D_N (N is the number of points of the periodic orbit) induced by the braid $\theta(\alpha)$. The normal dissection $\{B_P;\mathcal{R}\}$ is the image under f_θ of the trivial dissection of axis $\mathcal{R}$ and marked points the points of P. The normal dissection $\{F(B_P);\mathcal{R}\}$ is then the image by the homeomorphism $F\circ f_\theta$ of the trivial dissection of axis $\mathcal{R}$. By lemma 1.14, we have $\alpha_B(\alpha)=\theta(\alpha)\gamma_0$, where $\overline{\gamma_0}$ is ambient isotopic in M_β^3 to a suspension of P under $(F_t^\beta)_{t\in\mathbf{R}}$. Indeed, γ_0 is the braid corresponding to the periodic orbit of the homeomorphism F, which transforms $\{B_P;\mathcal{R}\}$ to $\{F(B_P);\mathcal{R}\}$. $\diamondsuit$

Proposition 3.19 *Let α be a braid with one component carried by $W_{(\psi,\Gamma)}$ and let $\alpha_S(\alpha)$, $\theta(\alpha)$ be the braids given respectively by lemma 3.17 and proposition 3.18.
Then $\alpha_S(\alpha)=\theta(\alpha)^{-1}$.*

Proof: In lemma 3.17-3, we proved that the braid $\alpha_S(\alpha)$ satisfies the following property: All the crossings are of the form $s_{i,j}^{-1}$, $i>j$. Moreover, a crossing-letter $s_{i,j}^{-1}$ occurs in $w(\alpha_S(\alpha))$ if and only if the position of the points i and j is reversed from the axis $\mathcal{R}$ to the axis A.

In order to prove proposition 3.19, it suffices to check:

1. All the crossings in $\theta(\alpha)$ have the form $s_{i,j}$ with $i<j$.
2. The crossing letter $s_{i,j}$ belongs to $w(\theta(\alpha))$ if and only if the position of the points i and j do not agree with their labelling along A.

Let us recall that we have projected the points of P on the accordion curve to obtain a collection of points P'. We have considered an axis A', passing through the points of P' in the order of their projection on A along the ties. This axis A' is homotopic to A, relative to ∂D^2. The projections on A, along the ties of $N(\Gamma)$, of the points in P are the terminal points of the paths B_P.

A path P_i in B_P does not intersect $\mathcal{R}$ in the rectangles it passes through, except perhaps in the last one which contains the corresponding point of the periodic orbit.
This comes from the construction of the accordion curve $\mathcal{R}$ and the fact that the paths P_i are parallel to the generating paths (see section 2 - definition 2.8 and lemma 2.12).

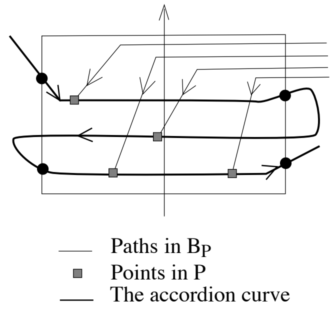


Figure 17: A non trivial dissection $\{B_P; \mathcal{R}\}$

The ordering of the intersection points $P_i \cap \mathcal{R}$ along P_i agrees with their ordering along $\mathcal{R}$. This is illustrated by figure 17.
Let us explain this figure. Recall that we have parametrized the rectangles $R(e)$ ($e \in E(\mathcal{T})$ by $[0,1] \times [-1,1]$, with $e \cap R(e) = [0,1] \times \{0\}$.
The paths P_i are decreasing in both directions $\{x\} \times [-1,1]$ and $[0,1] \times \{y\}$ in the rectangle $R(e)$. Indeed, the paths P_i are parallel to the generating paths. By definition 2.8, the generating paths satisfy the above property.
By definition of the accordion curve (see lemma 2.12), the ordering of the intervals $\varphi_s(\Gamma_s) \cap R(e)$, $e \in E(\mathcal{T})$, in $\mathcal{R}$ agrees with the ordering $\prec_e$ (see definition 1.1).

Therefore, all the crossing letters reconstructed from the normal dissection $\{B_P; \mathcal{R}\}$ are positive, i.e. of the form $s_{i,j}$, with $i < j$.
Indeed, all the return points of the reconstruction algorithm are on the right of the corresponding band (see section 1.3.2 - the reconstruction algorithm). This proves item 1).
Item 2) is proved by the same argument as in lemma 3.17 3-ii). $\diamondsuit$
Let us now complete the proof of lemma 3.13.
Since α is carried by the branched surface, then, by lemma 3.17, $\alpha = \alpha_B(\alpha)\alpha_S(\alpha)$. By propositions 3.18 and 3.19, $\alpha = \theta(\alpha)\gamma_0\theta(\alpha)^{-1}$ where $\overline{\gamma_0}$ is ambient isotopic in M^3_β to the suspension of P under $(F^\beta_t)_{t\in\mathbf{R}}$.

This completes the proof of lemma 3.13. $\diamondsuit$

Lemma 3.20 *The suspension under $(F^\beta_t)_{t\in\mathbf{R}}$ of any periodic orbit P of F is ambient isotopic in M^3_β to a unique closed braid carried by $\overline{W}_{(\psi,\Gamma)}$.*

Proof: The periodic orbit P defines a unique periodic admissible sequence for $\sum_{(\psi_s,\mathcal{T}_s)}$. By proposition 3.12, it gives a unique closed braid carried by $\overline{W}_{(\psi,\Gamma)}$. By lemma 3.13, this closed braid is ambient isotopic in M^3_β to the suspension under $(F^\beta_t)_{t\in\mathbf{R}}$ of a periodic orbit P' of F. The uniqueness, up to shift, of the symbolic codings implies that $P = P'$. $\diamondsuit$
This completes the proof of theorem 3.8. $\diamondsuit$

3.3 Coding the periodic orbits of the suspension flow

We suppose that a pseudo-Anosov braid $\beta \in \mathcal{B}_N$ and a canonical efficient representative (ψ, Γ) of $F \in [f_\beta]$ ($[f_\beta] \in MCG(D_N)$) are given. We denote by $\sum_{(\psi_s,\mathcal{T}_s)}$ the symbolic coding coming from the subdivided representative (ψ_s, Γ_s) (see definition 3.1 and below).
We recall that $(F^\beta_t)_{t\in\mathbf{R}}$ denotes the suspension flow of F associated to β.

Lemma 3.21 *Let $s_1, \cdots, s_k$ be a finite collection of admissible periodic sequences for $\Sigma_{(\psi_s,\mathcal{T}_s)}$ of respective lengths $l_1, \cdots, l_k$. Then there is an effective algorithm which gives a braid $\theta(s_1, \cdots, s_k)$ in $\mathcal{B}_{l_1+\cdots+l_k}$ whose closure has k components and is the suspension, under $(F_t^\beta)_{t\in\mathbf{R}}$, of the k periodic orbits corresponding to the sequences $s_1, \cdots, s_k$.*
In particular, one can compute the linking number of any pair of periodic orbits.

The strategy for proving this lemma is to use the braided branched surface $\overline{W}_{(\psi,\Gamma)}$ given by definition 3.7. Indeed, from theorem 3.8, this branched surface carries the suspension of all the regular periodic orbits. Once an admissible periodic sequence is given, one knows that the corresponding braid goes through an ordered sequence of bands of the branched surface. From the embedding of the branched surface $W_{(\psi,\Gamma)}$, one knows the crossings of the bands in $D^2 \times [0, 1-\epsilon]$ (for a small $\epsilon > 0$), and furthermore the identification of the intervals in $D^2 \times \{1\}$ given by definition 3.7 tells us which band crosses over which other band in $D^2 \times [1-\epsilon, 1]$. In order to complete the description of the braid corresponding to a given admissible periodic sequence, we only have to order the points in each interval of the branch locus. This ordering process will also work for a collection of periodic orbits. This process of ordering some points of an orbit along an interval is reminiscent to the kneading theory developped for continuous map of the interval (see for instance [CE]).

Definition 3.22 Let x be a regular periodic point for F and $(e_{k_1} \cdots e_{k_m})$ be the associated admissible sequence.

- We call *itinerary of* x the sequence: $I(x) = k_1^{\epsilon_1} \cdots k_m^{\epsilon_m}$ where $\epsilon_i = +1$ if $\psi_s(e_{k_i}) = \ldots e_{k_{i+1}} \ldots$ and $\epsilon_i = -1$ if $\psi_s(e_{k_i}) = \cdots e_{k_{i+1}}^{-1} \cdots$,

  $I_m(x)$ denotes the symbol k_m $(m \geq 1)$.

  Let x and x' be two periodic points for the homeomorphism F.

- We say that $I(x) \neq I(x')$ if there exists a positive integer k such that $I_k(x) \neq I_k(x')$.

- We will write $I_m(x) < I_j(x')$ if the interval on the initial axis of $W_{(\psi,\Gamma)}$ corresponding to $I_m(x)$ is distinct and preceds the interval corresponding to $I_j(x')$ according to the orientation of the axis.

- We define an order $\prec$ on the itineraries of the periodic points of F as follows:

  If $I(x) \neq I(x')$,

  Let m be the smallest integer such that $I_m(x) \neq I_m(x')$, then

  $I(x) \prec I(x')$ if:

  - There is an even number of $\epsilon_i = -1$ in the sequence $k_1^{\epsilon_1} \cdots k_{m-1}^{\epsilon_{m-1}}$ and $I_m(x) < I_m(x')$.
  - There is an odd number of $\epsilon_i = -1$ in the sequence $k_1^{\epsilon_1} \cdots k_{m-1}^{\epsilon_{m-1}}$ and $I_m(x') < I_m(x)$.

Proposition 3.23 *$I(x) \prec I(x')$ implies x preceds x' along the axis.*

The proof is the same than the one for continuous maps of the interval by the kneading theory. (see for instance [CE]). $\diamondsuit$
Let us recall (see lemma 3.17) that any braid α carried by the branched surface decomposes as $\alpha_B(\alpha)\alpha_S(\alpha)$, where $\alpha_B(\alpha)$ has a block partition. The corresponding crossing-word is obtained from a word in the block letters of the supporting braid (see lemma 3.17 1)). The

crossing-word for $\alpha_S(\alpha)$ is given by the lemma 3.17 once we know the respective positions of the points of the periodic orbit along the axis A and the accordion curve $\mathcal{R}$. The position of the points along A is computed from their itineraries by using the proposition 3.23. The position of the points along $\mathcal{R}$, i.e. the terminal axis of the supporting braid, is given by the permutation induced by $\alpha_B(\alpha)$.
One can now make explicit the algorithm announced in lemma 3.21. This will complete the proof of lemma 3.21, as well as the proof of the main theorem 0.1. We state it here for two periodic admissible sequences and at the end we compute their linking number.

1. `Find an efficient canonical representative` (ψ, Γ) `for` $\beta \in \mathcal{B}_N$

    - We apply the train-track algorithm (see [BH1], [FM], [Lo1]) starting from any induced automorphism $\varphi_{\#} : \pi_1(D_N) \to \pi_1(D_N)$.
    - We embed the graph Γ un D_N so that (ψ, Γ) is canonical. This step is achieved by a conjugacy but the explicit conjugacy is not necessary. We just have to fix the embedding so that the properties of definition 2.8 and proposition 2.3 are satisfied.

    Datas: two admissible periodic sequences σ_1, σ_2 for $\sum_{(\psi_s, \mathcal{T}_s)}$

2. `Construction of the special braided branched surface`

    (a) *Construction of the supporting braid* $\gamma(\mathcal{C})$
    We apply the supporting braid algorithm (see section 2).

    (b) *Construction of* $\overline{W}_{(\psi,\Gamma)}$.
    Follow definition 3.7.

3. Order all the points of the periodic orbits given by σ_1 and σ_2 along the branch locus of $W_{(\psi,\Gamma)}$.

    This is done by applying proposition 3.23 and comparing the itineraries.

4. Compute the braid α corresponding to the union of the two periodic orbits and also each individual braid α_1, α_2.

    To this end, we apply lemma 3.17 for the two orbits.

5. The closure of α gives rise to a two components link in M^3_β from which we compute the linking number.

An example

Let us start by a braid $\beta = \sigma_1\sigma_2^{-1}$ in $\mathcal{B}_3$. Its induced isotopy class is pseudo-Anosov and a canonical efficient representative (ψ, Γ) is shown in figure 18. The graph Γ has six edges, labelled a, b, c, 1, 2, 3, where the numbers correspond to boundary edges (see definition 1.6). Figure 19 a) shows the G-points and the generating paths for this example. Figure 19 b) shows the accordion curve $\mathcal{R}$, the nailed points and the images under φ (the embedding associated to (ψ, Γ) - see definition 1.3) of some generating paths, namely the one corresponding to the boundary paths (see section 4). Let us explain how we get these figures from the efficient representative (ψ, Γ). First, one gets the nailed points as the intersection points of $\varphi(\Gamma)$ with the sides of the polygons, which do not intersect the boundary edges (see definition 2.4). Then, we get the G-points as the pre-images under φ of the nailed points. Observe in particular that each extremity of a segment $e \cap R(e)$, $e \in \{a, b, c\}$, is a G-point. From this, we get the normal dissection $\mathcal{C}$ and the supporting braid $\gamma(\mathcal{C})$ shown in figure 20. The axis of the normal dissection $\mathcal{C}$ of this figure is the accordion curve of figure 19 b). In other words, we have drawn $\mathcal{R}$ flat, oriented from left to right and such that the initial points of the arcs of $\mathcal{C}$ belong to the top of the rectangle. These arcs correspond to the images under φ of the generating paths. In this figure the bolded arcs correspond to the boundary paths. The intersection points of these paths with $\mathcal{R}$ are represented by gray squares in figure 19 b). At these points, for reconstructing the normal dissection $\mathcal{C}$, we take care of the intersection signs. This means that we take care of the fact that the paths are oriented from above to below the axis, or the converse. From these informations, we recover the normal dissection of figure 20 by connecting all the intersection points by well-chosen disjointly embedded arcs. Then, we get the braid of figure 20 by the reconstruction algorithm (see section 1.3.2). The bolded strands form in fact the original braid β. Finally, one obtains the branched surface $W_\beta \equiv W_{(\psi,\Gamma)}$ (see figure 21) from the supporting braid $\gamma(\mathcal{C})$ of figure 20 as described in definition 3.7. All the intervals between two nailed points on the axis, which correspond to intervals lying in a same rectangle, are identified, respecting the transversal ordering. Here, from the choices we made, when one identifies two intervals, the one on the right is identified **over** the other interval. Figure 22 shows an example of a periodic orbit carried by W_β. The interested reader can experiment to get the symbolic coding of this periodic orbit.

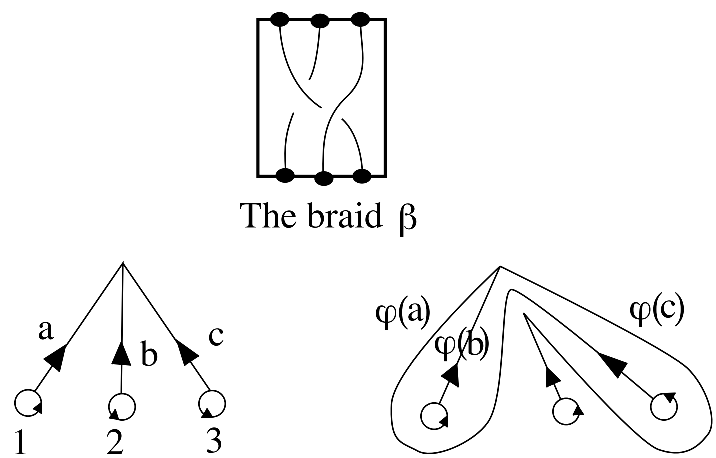


Figure 18: An efficient representative in D_3

$\psi(a) = ca^{-1}1^{-1}a \qquad \psi(b) = a \qquad \psi(c) = bc^{-1}3c$

$\psi(1) = 3 \qquad \psi(2) = 1 \qquad \psi(3) = 2.$

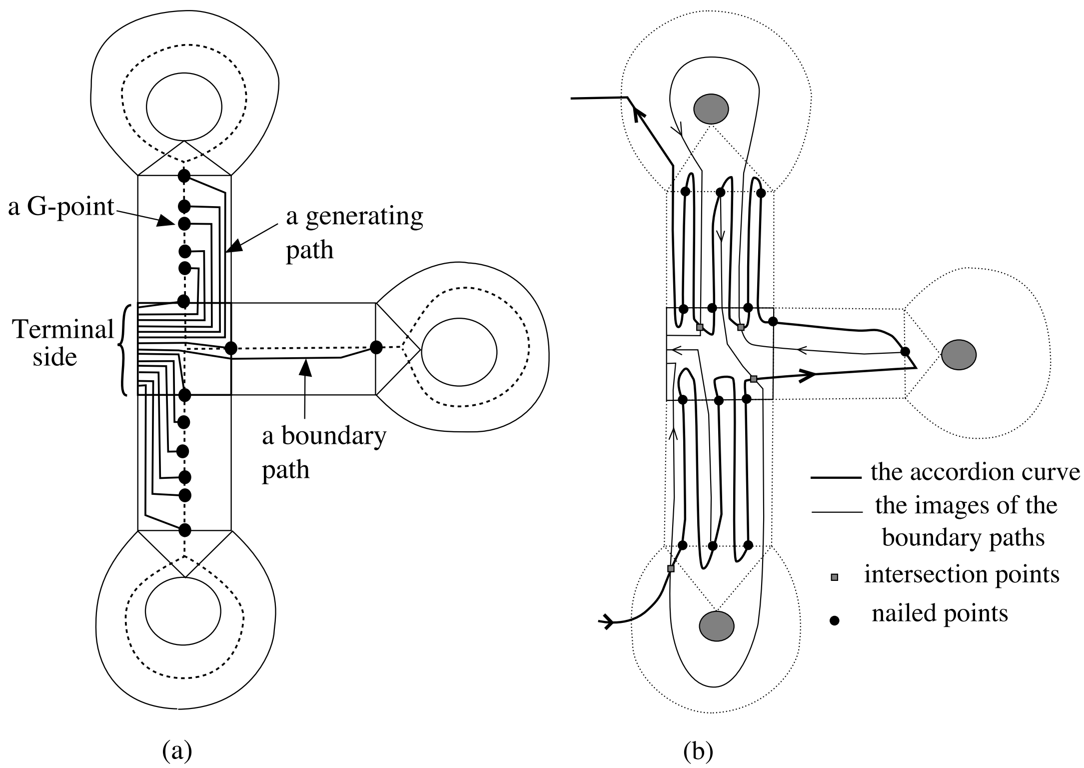


Figure 19: Generating paths and a part of the normal dissection $\mathcal{C}$

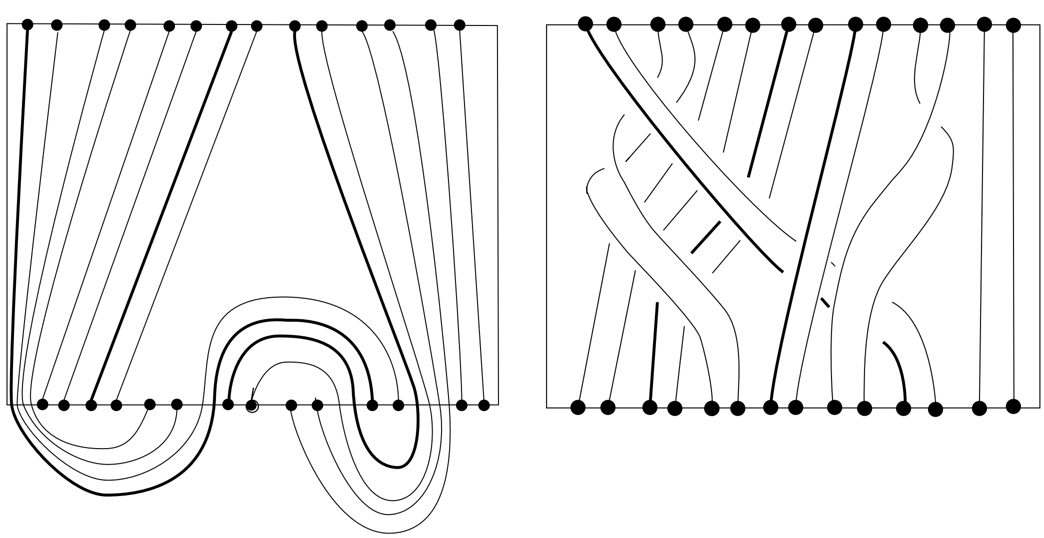


Figure 20: The normal dissection and the supporting braid

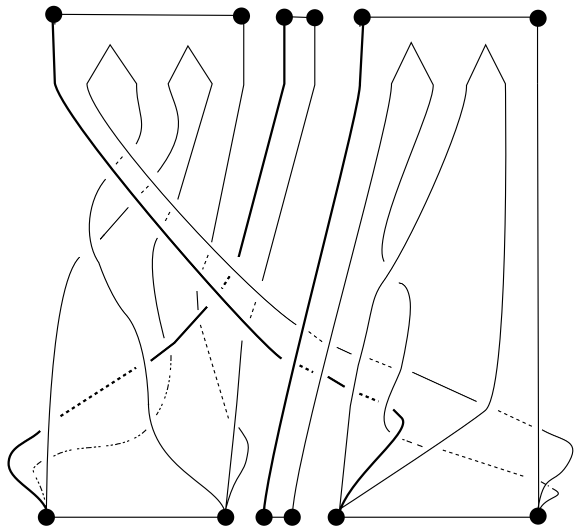


Figure 21: The branched surface $W_\beta \equiv W_{(\psi,\Gamma)}$

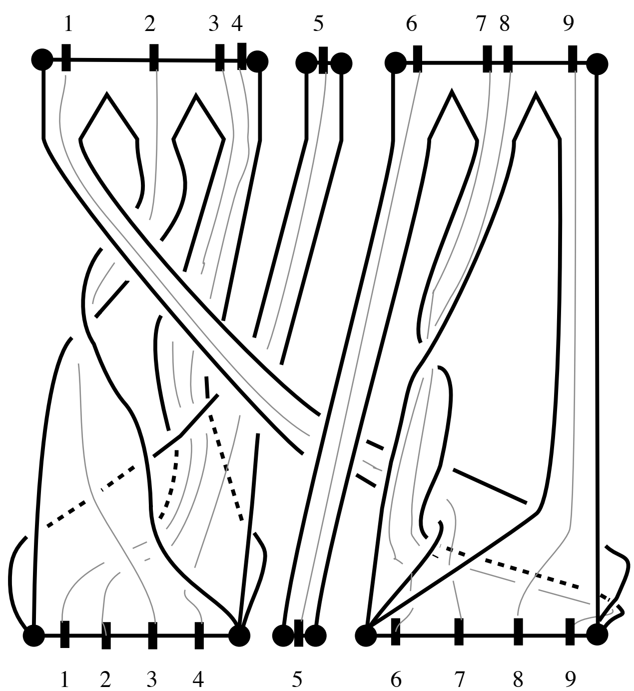


Figure 22: A braid α carried by W_β

4 Remarks and questions

Singular periodic orbits: As we already observed in section 3.1, the branched surface $\overline{W}_{(\psi,\Gamma)}$ that we construct by our algorithm carries the suspension of all the periodic orbits of the pseudo-Anosov homeomorphism, up to the finite collection formed by the periodic orbits, under ψ, of the vertices of the graph. For each such periodic orbit P_{sing} that we call *singular*, there is a boundary component of $\overline{W}_{(\psi,\Gamma)}$ which is a *cabling* of the suspension of P_{sing}. The corresponding braid in the cylinder $D^2 \times [0,1]$ is a "cabling" (as a braid) of the braid associated to P_{sing}.

Reconstruction algorithm: Our strategy for the construction of the branched surface $W_{(\psi,\Gamma)}$ is based on the construction of the supporting braid of section 2.2. An observation is that the reconstruction of this supporting braid and thus of $W_{(\psi,\Gamma)}$ can be realized knowing only the intersection points of the boundary paths (see section 2.2.1) with the accordion curve $\mathcal{R}$ (see lemma 11), their ordering along $\mathcal{R}$ and along each boundary path. The accordion curve $\mathcal{R}$ is the axis of the normal dissection $\mathcal{C}$ which gives the supporting braid. The intersection points of the other arcs of $\mathcal{C}$ with the axis $\mathcal{R}$ are then uniquely determined from the boundary paths. This affirmation is a corollary of lemma 2.22 and allows to reconstruct the supporting braid with less datas.

An easy reconstruction process: Our reconstruction process is defined for any topological representative (ψ, Γ) of a homeomorphism of the punctured disc, where Γ is canonical (see definition 1.6). Our aim was to apply it to an efficient representative of a pseudo-Anosov homeomorphism in order to obtain a branched surface which carries the best possible dynamics. If one only desired to reconstruct a branched surface associated to the suspension of some homeomorphism in the isotopy class, one could simplify the reconstruction process by choosing a particularly simple topological representative. This can be done for instance in the following way:

Let β be a representative of a braid in $\mathcal{B}_N$. One builds a normal dissection from β by sliding, at each crossing, the overcrossing strand along the undercrossing one until the crossing is supressed (for more details see [BZ]).

One identifies all the initial points of the normal dissection so constructed to a single one.

One applies an isotopy to the arcs to obtain a canonical topological representative (ψ, Γ) where Γ has exactly one valency N vertex v_0 and N boundary vertices. The vertex v_0 is fixed under ψ.

In this case, the construction of the supporting braid and thus of the branched surface is easy. A lot of choices of our general construction are natural (base point, escape paths,...) and in particular for the construction of the accordion curve $\mathcal{R}$.

The branched surface reconstructed by this process is not associated to a suspension flow of the pseudo-Anosov homeomorphism, but only of some homeomorphism in the isotopy class.

Universal template: As observed by the referee, we check that the branched surface W_β obtained from our construction in the above example, and embedded in $\mathbf{S}^3$, contains the universal template of Ghrist ([Gh]) as a subtemplate. In order to check this observation, we first embed our branched surface in $\mathbf{S}^3$ and isotope it to the template represented by figure 23. From this picture, we observe that, by removing the bands labelled by A and B, the template obtained is the universal template.

Since this universal template has the property of carrying any link in $\mathbf{S}^3$ and together with the results about the forcing relation in [Lo2], we get the following corollary. By *suspension flow* we mean a flow which is the suspension of a disc homeomorphism.

Corollary 4.1 *If a suspension flow in the solid torus admits a periodic orbit of type β given by the example, then any knot in $\mathbf{S}^3$ is represented by a periodic orbit of this flow.*

Question: Is there a criterion for a Pseudo-Anosov braid to "force" all knots in $\mathbf{S}^3$?

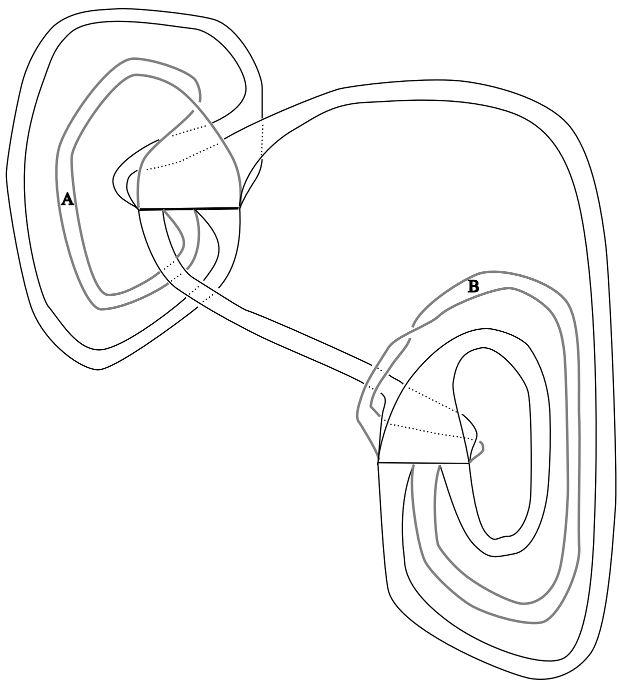


Figure 23: From W_β to the universal template

Acknowledgements: We would like to thank the referee for several interesting observations on our first version of this paper. In particular the remarks on the universal template are due to the referee.

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